1. Revisiting the DFT basis

In lecture, we show that the Discrete Fourier Fransform (DFT) represents a projection of a length n signal $\vec{x}$ onto the set of n sampled complex sinusoids generated by the n-th roots of unity. Here we want to discuss interpolation over DFT basis.

Recap of DFT: We can think of a real-world signal that is a function of time $x(t)$. By recording its values at regular intervals, we can represent it as a vector of discrete samples $\vec{x}$, of length n.

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[n-1] \end{bmatrix} \tag{1}$$

Let $\vec{X} = \begin{bmatrix} X[0] & \dots & X[n-1] \end{bmatrix}^T$ be the signal $\vec{x}$ represented in the frequency domain, that is

$$\vec{X} = U^{-1}\vec{x} = U^*\vec{x} \tag{2}$$

where U is a matrix of the DFT basis vectors ($\omega = e^{i\frac{2\pi}{n}}$).

$$U = \begin{bmatrix} | & & | \\ \vec{u}_0 & \cdots & \vec{u}_{n-1} \\ | & & | \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \tag{3}$$

Alternatively, we have that $\vec{x} = U\vec{X}$ or more explicitly

$$\vec{x} = X[0]\vec{u}_0 + \cdots + X[n-1]\vec{u}_{n-1} \tag{4}$$

In other words, $\vec{x}$ is a linear combination of the complex exponentials $\vec{u}_i$ with coefficients $X[i]$.

(a) For $x(t) = e^{i\frac{2\pi}{3}}$, sketch the real and complex parts versus t.

(b) Sample $x(t)$ at $t = 0, 1, 2$. How are those sampling points related to our DFT basis for $n = 3$?

(c) Show that for any n and k, $\vec{u}_{-k} = \vec{u}_{n-k}$

(d) According to (c), the DFT matrix can be represented as

$$U = \begin{bmatrix} | & | & | & & | & | \\ \vec{u}_0 & \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_{-2} & \vec{u}_{-1} \\ | & | & | & & | & | \end{bmatrix}$$

(e) Compute the DFT coefficients $\vec{X}$ for the following signal:

$$\vec{x} = \begin{bmatrix} \sin\left(\frac{2\pi}{3}(0)\right) & \sin\left(\frac{2\pi}{3}(1)\right) & \sin\left(\frac{2\pi}{3}(2)\right) & \sin\left(\frac{2\pi}{3}(3)\right) & \sin\left(\frac{2\pi}{3}(4)\right) & \sin\left(\frac{2\pi}{3}(5)\right) \end{bmatrix}^T.$$

(f) Now consider a real discrete-time signal:

$$\vec{x} = \begin{bmatrix} 1 \\ x[1] \\ x[2] \\ 1 \\ 1-\frac{\sqrt{3}}{2} \\ x[5] \end{bmatrix}^T.$$

We know $X[m] = 0$ for $m = \pm 2$. Show that we could recover the whole signal $\vec{x}$ based on the information above. What is $\vec{X}$?

(g) What if we don't know $x[4]$ is $1 - \frac{\sqrt{3}}{2}$? Is $\vec{x}$ unique?

(h) Given a continues time sinusoidal signal $x(t) = \sin\left(\frac{2\pi}{3}t\right)$, what is its frequency? What is the sampling rate for creating the discrete signal $\vec{x}$ in (e)?

(i) Sample $x(t) = \sin\left(\frac{2\pi}{3}t\right)$ with the sample rate = 2Hz between $0 \le t < 3$. How many data points do you get? Collect those sample points as a discrete signal $\vec{y}$. Compare the DFT coefficients of $\vec{y}$ with the result in (e). Explain their relationship.

(j) Sample $x(t) = \sin\left(\frac{2\pi}{3}t\right)$ with the sample rate = 2Hz between $0 \le t < 6$. How many data points do you get? Collect those sample points as a discrete signal $\vec{z}$. Compare the DFT coefficients of $\vec{z}$ with the result in (e). Explain their relationship.

(k) Consider a length n discrete-time signal $\vec{x}$, along with its DFT coefficients, $\vec{X}$. If we know $X[m] = 0$, for all $|m| > k$, what is the minimum number of sampling points we need to interpolating the $\vec{x}$?

(l) Sample $x(t) = \sin\left(\frac{2\pi}{3}t\right)$ with the sample rate = 2/3 Hz between $0 \le t < 3$. Are you able to reconstruct the signal based on the sample points?