1. Sampling in control systems

Consider a linear continuous-time scalar control system:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

We can sample it every T seconds and get a discrete-time form of the control system. The discretization of the state equations is a sampled discrete time-invariant system given by

\[ x_d(k+1) = A_dx_d(k) + B_du_d(k) \]  

(1)

Here, \( x_d(k) \) denotes \( x(kT) \). This is a snapshot of the state. The relationship between the discrete-time input \( u_d(k) \) and the actual input applied to the physical continuous-time system is that \( u(t) = u_d(k) \) for all \( kT \leq t < (k+1)T \).

(a) Argue intuitively why if the continuous-time system is stable, the corresponding discrete-time system should be stable too. Similarly, argue intuitively why if the discrete-time system is unstable, then the continuous-time system should also be unstable. Answer: If the continuous-time system is bounded with all bounded inputs, then a bounded discrete-time input is also bounded when realized in continuous time and hence results in a bounded continuous-time state. Sampling this state would stay bounded.

Similarly, if there is a bounded input that would send the discrete-time system spiraling out to infinity, then that particular input is also valid in continuous-time since it is realized in a piecewise constant way. It sends the sampled state out of bounds and so must not be bounded in continuous-time either.

(b) In the scalar case \( A \) and \( B \) are just constants. What are the new constants \( A_d \) and \( B_d \)?

(HINT: Think about solving this one step at a time. Everytime a new control is applied, this is a simple differential equation with a new constant input. How does \( \dot{x}(t) = \lambda x(t) + u \) evolve with time if it starts at \( x(0) \)? Notice that \( x(0)e^{\lambda t} + \frac{1}{\lambda} (e^{\lambda t} - 1) \) seems to solve this differential equation.)

Answer: \( A_d = e^{\lambda T} \) and \( B_d = \frac{e^{\lambda T} - 1}{\lambda} B \).

2. Sampling rate vs DFT

In this question, we want to discuss how the sampling rates and the number of samples taken influence the frequency domain representation of the sampled signal. Also, we like to discuss the DFT basis with different \( n \).

Remember we have that \( \vec{x} = U \vec{X} \) or more explicitly

\[ \vec{x} = X[0] \vec{u}_0 + \cdots + X[n-1] \vec{u}_{n-1} \]  

(2)

That is, \( \vec{x} \) is a linear combination of the (normalized) complex exponentials \( \vec{u}_i \) with coefficients \( X[i] \).
(a) Compute the DFT coefficients $\tilde{X}$ for the following signal:

$$\tilde{x} = [\sin(\frac{2\pi}{3}(0)) \quad \sin(\frac{2\pi}{3}(1)) \quad \sin(\frac{2\pi}{3}(2))]^T.$$ 


(b) Given a continuous time sinusoidal signal $x(t) = \sin(\frac{2\pi}{3}t)$, what is its frequency? What is the sampling rate under which taking three samples would give rise to the finite vector of samples $\tilde{x}$ in (a)?

(HINT: The second part of this is a trick question.)

**Answer:** Given a signal, the frequency means the rate at which it is repeated over a particular period of time. The continues time signal $x(t) = \sin(\frac{2\pi}{3}t)$ repeats once for three seconds. The frequency is 1/3, so our sampling rate should be higher than 2/3. One possible sampling rate for (a) is 1 Hz. The signal in (a) can also be given by sampling $\sin(\frac{2\pi}{3}t)$ with the sampling rate = 1 Hz choosing $t = 0, 4$ and 8. Notice that $\sin(\frac{2\pi}{3}(4)) = \sin(\frac{2\pi}{3}(3+1)) = \sin(2\pi + \frac{2\pi}{3}(1)) = \sin(\frac{2\pi}{3}(1))$.

(c) Sample $x(t) = \sin(\frac{2\pi}{3}t)$ at 1Hz between 0 ≤ $t$ ≤ 6. How many data points do you get? Collect those sample points as a finite vector/signal $Y$. Compare the DFT coefficients of $\tilde{Y}$ with what you had gotten in (a). Explain their relationship.

**Answer:** This is the same question as the one in Dis13A.

$$\tilde{y} = [\sin(\frac{2\pi}{3}(0)) \quad \sin(\frac{2\pi}{3}(1)) \quad \sin(\frac{2\pi}{3}(2)) \quad \sin(\frac{2\pi}{3}(3)) \quad \sin(\frac{2\pi}{3}(4)) \quad \sin(\frac{2\pi}{3}(5))]^T.$$ 

There are 6 points. Here the $m$-th entry of the signal is $\sin(\frac{2\pi}{3}(m)) = e^{i\frac{2\pi}{3}m}e^{-i\frac{2\pi}{3}(2m)} = \sqrt{6}(\sqrt{3}[-u_2[m]-u_2[m]])$. Therefore, $X[2] = \frac{\sqrt{6}}{2}$, $X[4] = -\frac{\sqrt{6}}{2}$, while others are zero. Ideally this signal is the same as the one in (a), but with different sampling points, we will project it into a different set of sampled sinusoids(with different scaling factors: $\sqrt{3}$ and $\sqrt{6}$).

(d) Sample $x(t) = \sin(\frac{2\pi}{3}t)$ at 2Hz between 0 ≤ $t$ ≤ 3. How many data points do you get? Collect those sample points as a finite vector/signal $\tilde{Z}$. Compare the DFT coefficients $\tilde{Z}$ with the result in (a). Explain their relationship.

**Answer:** 6 samples:

$$\tilde{y} = [\sin(\frac{2\pi}{6}(0)) \quad \sin(\frac{2\pi}{6}(1)) \quad \sin(\frac{2\pi}{6}(2)) \quad \sin(\frac{2\pi}{6}(3)) \quad \sin(\frac{2\pi}{6}(4)) \quad \sin(\frac{2\pi}{6}(5))]^T.$$ 

The DFT coefficients will be zeros except $Y[1] = \frac{\sqrt{6}}{2}$ and $Y[5] = Y[-1] = -\frac{\sqrt{6}}{2}$.

(e) What if we sampled $x(t) = \sin(\frac{2\pi}{3}t)$ with the sample rate = 1Hz between 0 ≤ $t$ < 3T, where $T$ is a natural number. How many data points would you get? How will the corresponding DFT basis look like? How would you imagine the DFT coefficients looking like?

**Answer:** Notice that we revise the description here. The number of samples will be 3T. As in (c), the $m$-th entry of the signal is $\sin(\frac{2\pi}{3}(m)) = e^{i\frac{2\pi}{3}(m)}e^{-i\frac{2\pi}{3}(2m)} = \sqrt{3}(\sqrt{3}[-u_2[m]-u_2[m]])$. Therefore, $X[T] = \frac{\sqrt{6}}{2}$, $X[-T] = -\frac{\sqrt{6}}{2}$, while others are zero. Discuss the condition when $T$ is getting bigger and bigger.

(f) Alternatively, think about the case of sampling $x(t) = \sin(\frac{2\pi}{3}t)$ at rate = $n_s$Hz between 0 ≤ $t$ < 3, where $n_s \rightarrow \infty$. What is the sampling period? How many data points would you get? How will the corresponding DFT basis look like? What would you imagine the DFT coefficients looking like?

**Answer:** Notice that we change the time period here. The sampling period is $\frac{1}{n_s}$, while the total number of samples is 3$n_s$. The DFT basis will be looked like the one in (c), for $n_s = T$. However, the frequency of the sampled signal under this sampling rate is $\frac{1}{3n_s}$, and the $m$-th entry of the signal is...
\[
sin\left(\frac{2\pi}{3n_s}(m)\right).
\]
Under the DFT basis with \( n = 3n_s \), \( \sin\left(\frac{2\pi}{3n_s}(m)\right) = e^{\frac{i2\pi m(n_s)}{2i}} - e^{-\frac{i2\pi m(n_s)}{2i}} = \sqrt{3n_s} (\vec{u}_{1}[m] - \vec{u}_{-1}[m]) \).

Therefore, \( X[1] = \sqrt{3n_s}, \ X[-1] = -\sqrt{3n_s} \), while others are zero. Discuss the condition when \( n_s \) is getting bigger and bigger.

\( (g) \) Consider a length \( n \) discrete-time finite vector/signal \( \vec{x} \), along with its DFT coefficients, \( \vec{X} \). If we know \( X[m] = 0 \), for all \( |m| > k \), what is the minimum number of sampling points we need to interpolate all of \( \vec{x} \)? **Answer:** 2\(k + 1\)

\( (h) \) (Optional) Extend the above idea to the continuous time case, what will be the minimum frequency of sampling to fully reconstruct a signal \( x(t) \), which includes no sinusoids with frequencies higher than \( f \)?

**Answer:** This is the sampling theorem: the sampling rate should be greater than 2\(f\). You could use the (f) to explain it.

\( (i) \) (Optional) Sample \( x(t) = \sin\left(\frac{2\pi t}{3T}\right) \) at rate 2/3 Hz between \( 0 < t < 3T \). Are you able to reconstruct the signal based on the sample points? Why or why not?

**Answer:** No. If you are doing this sampling at \( t = 0 \) and 3/2, you will get \([0 \ 0 \ \cdots]^T\). It is impossible to reconstruct the original sinusoidal signal. Here we need to use the concept of discrete-time Fourier transform (dtft), which takes infinite number of sample points.