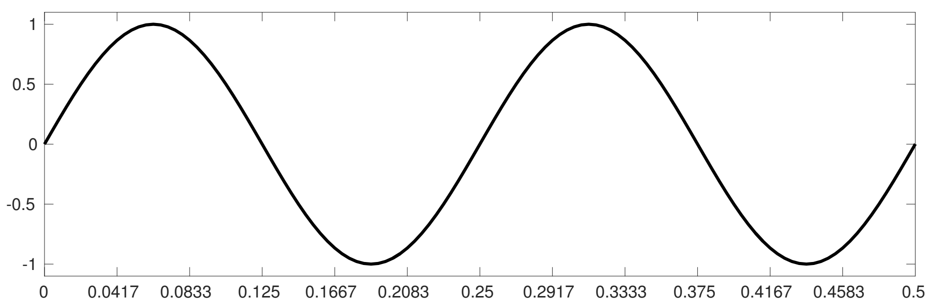
EECS 16B Designing Information Devices and Systems II
Spring 2016 Anant Sahai and Michel Maharbiz **Discussion 13B**

1. **The wagon wheel effect**

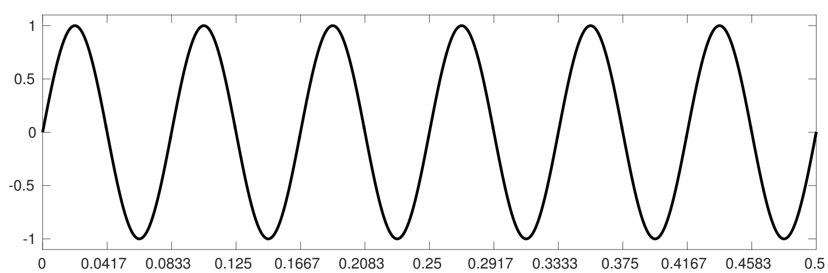
Suppose that we are recording a video of a wheel spinning. Our video camera captures an image of the wheel at a rate of 24 frames per second. Furthermore, let's assume that the wheel is spinning in the XY plane, and the camera is pointed down the Z axis such that it is looking at the wheel straight on. The wheel has a radius of 1 unit, and each image of the wheel is centered such that it can be represented as a plot of the unit circle about the origin of the XY plane.

(a) We want to track the y position of a point on the wheel through time based on what we see in the recorded video. The point starts at (1,0) at time $t = 0$. The wheel spins counter clockwise from the persepctive of the camera at a rate of 4 revolutions per second, and the y position of the point is given by $y(t) = \sin(8\pi t)$. Indicate on the plot below where we will sample the y position of the point from $t = 0$ to $t = 1/2$.



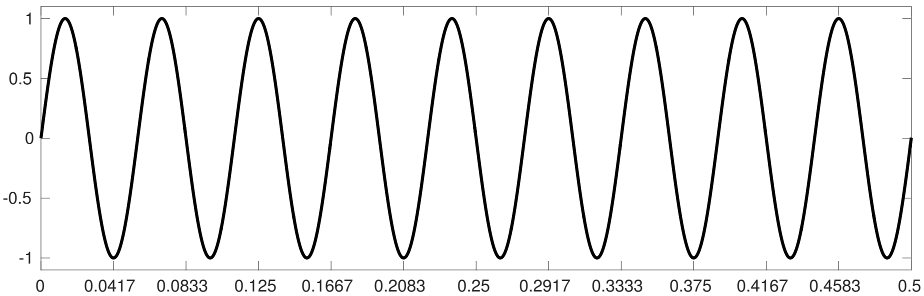
(b) What does this sampled signal look like in the frequency domain when we apply the DFT?

(c) Now let's suppose that the wheel spins at a rate of 12 revolutions per second. Indicate on th plot where we will sample the y position of the point.



(d) What does the sampled signal look like now in the frequency domain when we apply the DFT?

(e) Now, let's spin the wheel at 18 revolutios per second. Indicate where the y position is sampled.



(f) What does the sampled signal look like in the frequency domain now?

(g) How many frames per second would our camera need to record the wheel spinning at a rate of n revolutions per second so that we can accurately represent the y position of the point on the wheel over time in the frequency domain?

2. Sampling in control systems

Consider a linear continuous-time scalar control system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

We can sample it every T seconds and get a discrete-time form of the control system. The discretization of the state equations is a *sampled* discrete time-invariant system given by

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k) \tag{1}$$

Here, $x_d(k)$ denotes $x(kT)$. This is a snapshot of the state. The relationship between the discrete-time input $u_d(k)$ and the actual input applied to the physical continuous-time system is that $u(t) = u_d(k)$ for all $kT \leq t < (k+1)T$.

(a) Argue intuitively why if the continuous-time system is stable, the corresponding discrete-time system should be stable too. Similarly, argue intuitively why if the discrete-time system is unstable, then the continuous-time system should also be unstable.

(b) In the scalar case A and B are just constants. What are the new constants A_d and B_d?

(HINT: Think about solving this one step at a time. Everytime a new control is applied, this is a simple differential equation with a new constant input. How does $\dot{x}(t) = \lambda x(t) + u$ evolve with time if it starts at $x(0)$? Notice that $x(0)e^{\lambda t} + \frac{u}{\lambda}(e^{\lambda t} - 1)$ seems to solve this differential equation.)

3. Sampling rate vs DFT

In this question, we want to discuss how the sampling rates and the number of samples taken influence the frequency domain representation of the sampled signal. Also, we would like to discuss the DFT basis with different n.

Remember we have that $\vec{x} = U\vec{X}$ or more explicitly

$$\vec{x} = X[0]\vec{u}_0 + \cdots + X[n-1]\vec{u}_{n-1} \tag{2}$$

That is, $\vec{x}$ is a linear combination of the (nomalized) complex exponentials $\vec{u}_i$ with coefficients $X[i]$.

(a) Compute the DFT coefficients $\vec{X}$ for the following signal:

$$\vec{x} = \left[\sin\left(\frac{2\pi}{3}(0)\right) \quad \sin\left(\frac{2\pi}{3}(1)\right) \quad \sin\left(\frac{2\pi}{3}(2)\right)\right]^T.$$

(b) Given a continuous time sinusoidal signal $x(t) = \sin\left(\frac{2\pi}{3}t\right)$, what is its frequency? What is the sampling rate under which taking three samples would give rise to the finite vector of samples $\vec{x}$ in (a)?

(HINT: The second part of this is a trick question.)

(c) Sample $x(t) = \sin\left(\frac{2\pi}{3}t\right)$ at 1Hz between $0 \leq t < 6$. How many data points do you get? Collect those sample points as a finite vector/signal $\vec{y}$. Compare the DFT coefficients of $\vec{y}$ with what you had gotten in (a). Explain their relationship.

(d) Sample $x(t) = \sin\left(\frac{2\pi}{3}t\right)$ at 2Hz between $0 \leq t < 3$. How many data points do you get? Collect those sample points as a finite vector/signal $\vec{z}$. Compare the DFT coefficients $\vec{Z}$ with the result in (a). Explain their relationship.

(e) What if we sampled $x(t) = \sin\left(\frac{2\pi}{3}t\right)$ with the sample rate = 1Hz forever (between $0 \leq t < \infty$). How many data points would you get? How will the corresponding DFT basis look like? How would you imagine the DFT coefficients looking like?

(f) Alternatively, think about the case of sampling $x(t) = \sin\left(\frac{2\pi}{3}t\right)$ at rate = n_sHz between $0 \leq t < 1$, where $n_s \to \infty$. What is the sampling period? How many data points would you get? How will the corresponding DFT basis look like? What would you imagine the DFT coefficients looking like?

(g) Consider a length n discrete-time finite vector/signal $\vec{x}$, along with its DFT coefficients, $\vec{X}$. If we know $X[m] = 0$, for all $|m| > k$, what is the minimum number of sampling points we need to interpolate all of $\vec{x}$?

(h) Extend the above idea to the continuous time case, what will be the minimum frequency of sampling to fully reconstruct a signal $x(t)$, which includes no sinusoids with frequencies higher than f?

(i) Sample $x(t) = \sin\left(\frac{2\pi}{3}t\right)$ at rate 2/3 Hz between $0 \leq t < 3$. Are you able to reconstruct the signal based on the sample points? Why or why not?