1. Cosine Transformation

Assume that we are dealing with signals of length $n$.

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[n-1] \end{bmatrix} = \sum_{p=0}^{n-1} X[p] \vec{u}_p = \begin{cases} \sum_{p=-\frac{n}{2}}^{\frac{n}{2}} X[p] \vec{u}_p & \text{if } n \text{ odd} \\ \sum_{p=-\frac{n}{2}+1}^{\frac{n}{2}} X[p] \vec{u}_p & \text{if } n \text{ even} \end{cases}$$ (1)

In the lectures, we have seen that we can represent signals with sums of complex exponentials in the DFT basis. One non-intuitive aspect is that, even when the signal is real, the basis is complex. In this problem, we will explore a different representation in which real signals are written as linear combinations of periodic real signals.

Specifically, we will show how to derive

$$x[t] = \alpha_0 + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \alpha_m \cos \left( \frac{2\pi m}{n} t + \phi_m \right).$$ (2)

First, we need to understand cosines with phases.

(a) For a real signal $\vec{x}$, the DFT coefficients are conjugate symmetric, i.e. $X[m] = X[-m]^*$ (you will show this in the homework). Therefore, suppose $X[m] = re^{i\theta}$, what is $X[-m]$?

(b) Now assume that, for a real signal $\vec{x}$, its DFT coefficients are $X[m] = 0$ for $m \neq \pm 5$. Show that we can represent the $t$-th component of $\vec{x}$ by $x[t] = \alpha \cos \left( \frac{2\pi}{n} 5t + \phi \right)$. Find $\alpha$ and $\phi$. Your answer should be in terms of $|X[5]|$ and $\angle X[5]$.

(c) Therefore, let $\vec{x}$ be an arbitrary signal of length $n$, where $n$ is odd. Write it as a sum of cosines where the cosine scaling and phase are written in terms of the DFT coefficients, $X[-\frac{n-1}{2}], \ldots, X[\frac{n-1}{2}]$. You can use $\angle z$ and $|z|$ to refer to the angle and magnitude of a complex number, respectively (i.e. $z = |z|e^{i\angle z}$).

(d) How about when $n$ is even?

Answer:

(a) $X[-m] = re^{-i\theta}$.

(b) We can write out the signal by:

$$x[t] = \frac{2|X[5]|}{\sqrt{n}} \cos \left( \frac{2\pi}{n} 5t + \angle X[5] \right)$$

(c) A real signal has Hermitian symmetric coordinates; thus

$$x[t] = \frac{1}{\sqrt{n}} \left( X[0] + \sum_{m=1}^{\frac{n-1}{2}} 2|X[m]| \cos \left( \frac{2\pi mt}{n} + \angle X[m] \right) \right)$$
(d) When $n$ is even, we have a mismatched pair at $p = \frac{n}{2}$, which is equal to

$$\frac{1}{\sqrt{n}}X\left[\frac{n}{2}\right] e^{-\frac{2\pi in}{n}} = \frac{1}{\sqrt{n}}X\left[\frac{n}{2}\right] \cos(\pi t).$$

2. Phase response  Let $\vec{x}$ be a real signal of length $n$ (assume $n$ odd till the end). In the previous part, we showed that we could write

$$x[t] = \alpha_0 + \sum_{p=1}^{\frac{n}{2}} \alpha_p \cos\left(\frac{2\pi p}{n} t + \phi_p\right),$$

where there are $1 + \frac{n-1}{2}$ different $\alpha_p$ parameters and $\frac{n-1}{2}$ parameters $\phi_p$.

Let $C$ be a circulant matrix with eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$. In lecture, we have seen that the DFT basis diagonalizes $C$ (which correspond to LTI systems). However, the basis is complex and the eigenvalues are usually complex. However, if we push a real signal through $C$, we will get a real signal back. Where do all the imaginary parts go, then?

Let $\vec{y}$ be the output of $C$ with the input $\vec{x}$. The reason why the DFT basis is so useful is that, since $C$ is diagonalized by the basis, we have

$$y[t] = \frac{1}{\sqrt{n}} \sum_{p=0}^{n-1} \lambda_p X[p] e^{i\frac{2\pi p}{n} t} \quad (3)$$

(a) Use the fact that the complex exponentials are eigenvectors of $C$ to write out what the output of $C$ is when given the input $\vec{x}$, for the specific case of $x[t] = \cos\left(\frac{2\pi p}{n} t + \theta\right)$.

(b) Using the fact that the eigenvalues $\lambda_p$ for a real circulant matrix $C$ exhibit conjugate symmetry (from the Homework), what cosine does this output correspond to?

(c) What does the system $C$ do to cosines in terms of the effect on their magnitude, frequency, and phase?

(d) Write $y[t]$ entirely as a sum of cosines.

(e) What changes for $n$ even? (Think about what $\lambda_{\frac{n}{2}}$ must be like for a real $C$ matrix.)

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