1. **Geometric interpretation of the SVD** In this exercise, we explore the geometric interpretation of symmetric matrices and how this connects to the SVD. We consider how a real 2×2 matrix acts on the unit circle, transforming it into an ellipse. It turns out that the principal semiaxes of the resulting ellipse are related to the singular values of the matrix, as well as the vectors in the SVD.

   (a) Consider the real 2×2 matrix
   $$A = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}.$$
   Now consider the unit circle in $\mathbb{R}^2$,
   $$S = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mid 0 \leq \theta < 2\pi \right\}.$$
   What is the image of S under the action of A? Plot the image AS in $\mathbb{R}^2$.

   (b) We can think of the unit circle as being the level set of the function $\vec{x}^T\vec{x}$ for the level 1. In other words the solution to $\vec{x}^T\vec{x} = 1$ for a two-dimensional vector $\vec{x}$. Verify that the unit-circle satisfies this equation and then find the similar expression satisfied by AS.

   (c) Compute AA^T and A^TA and diagonalize them. Do you notice something about the eigenvalues? What about the eigenvectors?

   (d) Use the above to compute the Singular Value Decomposition: $A = U\Sigma V^T$. Here, we require that Σ be a diagonal matrix and U and V both be matrices whose columns are orthonormal vectors. By convention, we also require the diagonal entries of Σ to be sorted by magnitude, starting with the biggest.
   (Hint: plug in the candidate decomposition into AA^T and A^TA and see what you notice.)

   (e) Consider the columns of the matrices U, V obtained in the previous part, and treat them as vectors in $\mathbb{R}^2$. Let $U = (\vec{u}_1\, \vec{u}_2)$, $V = (\vec{v}_1\, \vec{v}_2)$. Let σ_1, σ_2 be the singular values of A, where $\sigma_1 \geq \sigma_2$.
   Draw in your plot of AS the vectors $\sigma_1\vec{u}_1$ and $\sigma_2\vec{u}_2$, drawn from the origin. What do these vectors correspond to geometrically?

   (f) Repeat what you did above for the matrix $A = \begin{pmatrix} +1 & -1 \\ 2 & 2 \end{pmatrix}$.
   Do you notice something?

   (g) Consider the case where A is a real $n \times n$ symmetric matrix. What do you observe geometrically in this case?

Contributors:

- Lynn Chua.
- Shane Barratt.