1. Two Capacitors

Consider the circuit below, assume that when $t \leq 0$, both capacitors have no charge ( $V_1(t = 0) = 0$ and $V_2(t = 0) = 0$). At $t = 0$, the switch closes.

![Two Capacitor Circuit with Voltage Source](image)

(a) First, use Kirchoff’s Laws and the capacitor equation ($I = \frac{dV}{dt}C$) to find the differential equation of this circuit.

**Answer:**

$$\frac{d}{dt}\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} + \begin{bmatrix} \frac{V}{R_1C_1} \\ 0 \end{bmatrix}$$

(b) As shown in class, we can write each voltage $V(t)$ as $V_{std}(t) + V_{trans}(t)$, where $V_{std}(t)$ is the steady state function and $V_{trans}(t)$ is the transient function.

What are the steady state functions of $V_1(t)$ and $V_2(t)$?

**Answer:** $V_{1, std}(t) = V$ and $V_{2, std}(t) = V$

(c) Now, replace $V_1(t) = V_{1, std}(t) + V_{1, trans}(t)$ and $V_2(t) = V_{2, std}(t) + V_{2, trans}(t)$ using the steady state functions you found in the previous part. Also, find the initial conditions of $V_{1, trans}(t)$ and $V_{2, trans}(t)$ when $t = 0$. **Answer:** $V_{1, trans}(0) = -V$ and $V_{2, trans}(0) = -V$

$$\frac{d}{dt}\begin{bmatrix} V_{1, trans}(t) \\ V_{2, trans}(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$$

(d) Assume that $C_1 = C_2 = 1$, $R_1 = \frac{1}{3}$, and $R_2 = \frac{1}{2}$. Diagonalize the matrix $A$ in $\frac{d}{dt}\begin{bmatrix} V_{1, trans}(t) \\ V_{2, trans}(t) \end{bmatrix} = A\begin{bmatrix} V_{1, trans}(t) \\ V_{2, trans}(t) \end{bmatrix}$
Answer:  \[ A = U \Lambda U^T \]

\[ \Lambda = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \]

\[ U = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \]

(e) Now that we have diagonalized the matrix, we can now work in the eigenspace. Let us call the transformed \[ \begin{bmatrix} V_{1,\text{trans}}(t) \\ V_{2,\text{trans}}(t) \end{bmatrix} \] as \[ \tilde{V}(t) = \begin{bmatrix} \tilde{V}_1(t) \\ \tilde{V}_2(t) \end{bmatrix}. \] Solve for \[ \tilde{V}(t). \] Do not forget about the initial conditions of \[ \tilde{V}(t). \]

Answer: Initial Conditions:

\[ \tilde{V}(0) = \frac{V}{\sqrt{5}} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \]

\[ \tilde{V}(t) = \frac{V}{\sqrt{5}} \begin{bmatrix} e^{-6t} \\ e^{-t} \end{bmatrix} \]

(f) Now that we have \[ \tilde{V}(t), \] find the solution for \[ \begin{bmatrix} V_{1,\text{trans}}(t) \\ V_{2,\text{trans}}(t) \end{bmatrix}. \]

Answer:

\[ \begin{bmatrix} V_{1,\text{trans}}(t) \\ V_{2,\text{trans}}(t) \end{bmatrix} = \frac{V}{5} \begin{bmatrix} -2e^{-6t} - 3e^{-t} \\ e^{-6t} - 6e^{-t} \end{bmatrix} \]

(g) For the final step, solve for \[ \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}. \]

Answer:

\[ \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = V \begin{bmatrix} 1 - \frac{2}{5} e^{-6t} - \frac{3}{5} e^{-t} \\ \frac{1}{5} e^{-6t} - \frac{3}{5} e^{-t} \end{bmatrix} \]

(h) Sketch the voltage vs time plots of \[ V_1(t) \] and \[ V_2(t). \]

Answer: This assumes the \[ V = 1. \]
2. RC Circuit - NAND  

Let us consider the RC Circuits of a NAND logic gate. This circuit implements the boolean function $\neg(A \land B)$.

![Figure 2: NAND](image1.png)

![Figure 3: RC Model of NAND](image2.png)

As shown in the figure, we can replace all the transistors with a resistor and switch but there is also a
capacitor between the nMOS gates in the PUN. This capacitor complicates how long it takes for \( V_{out} \) to obtain the correct value. We will see how shortly.

Assume that \( A \) and \( B \) can only have two voltage values; \( V_{DD} \) (1) and Ground (0). \( A \) and \( B \) can change instantaneously.

(a) Assume for \( t < 0 \), \( A = 1 \) and \( B = 0 \) for a long enough time. What are the voltages across the two capacitors?

\[ V_C(0) = V_{out}(0) = V_{DD} \]

(b) At \( t = 0 \), the inputs change: \( A = 1 \) and \( B = 1 \). What is the resulting circuit of the NAND? Find the differential equation for \( V_{out} \).

\[ R_1 \]
\[ I_3(t) \]
\[ I_2(t) \]
\[ R_2 \]
\[ V_{in}(t) \]
\[ + \]
\[ - \]
\[ C \]
\[ V_{out}(t) \]
\[ C_L \]

\[ \frac{d}{dt} \left[ \begin{array}{c} V_C(t) \\ V_{out}(t) \end{array} \right] = \left[ \begin{array}{cc} -\frac{1}{R_1 C} - \frac{1}{R_2 C} & \frac{1}{R_2 C_L} \\ \frac{1}{R_1 C_L} & -\frac{1}{R_2 C_L} \end{array} \right] \left[ \begin{array}{c} V_C(t) \\ V_{out}(t) \end{array} \right] \]

(c) Assume that \( R_1 = \frac{1}{3} \), \( R_2 = 1 \), \( C = \frac{1}{2} \), and \( C_L = \frac{1}{3} \). Solve the differential equation in part (b).

\[ V_C(t) = \frac{V_{DD}}{7} \left[ 3e^{-2t} + 6e^{-9t} \right] \]

\[ V_{out}(t) = 6e^{-2t} - 2e^{-9t} \]

(d) Let us try another situation. Assume for \( t < 0 \), \( A = 0 \) and \( B = 1 \). At \( t = 0 \), the inputs change: \( A = 1 \) and \( B = 1 \). What is the resulting circuit of the NAND? What are the initial conditions for the voltages across the two capacitors? Find the differential equation for \( V_{out} \).

\[ \text{Answer: Initial Conditions: } V_C(0) = 0 \text{ and } V_{out}(0) = V_{DD} \]
\[
\frac{d}{dt} \begin{bmatrix} V_C(t) \\ V_{out}(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C} + \frac{1}{R_2C} \right) & \frac{1}{R_2C_L} \\ \frac{1}{R_1C} & -\frac{1}{R_2C_L} \end{bmatrix} \begin{bmatrix} V_C(t) \\ V_{out}(t) \end{bmatrix}
\]

(e) Assume that \( R_1 = \frac{1}{3}, R_2 = 1, C = \frac{1}{2}, \) and \( C_L = \frac{1}{5}. \) Solve the differential equation in part (d).

**Answer:** The solution of the differential equation is the following:

\[
\begin{bmatrix} V_C(t) \\ V_{out}(t) \end{bmatrix} = \frac{V_{DD}}{7} \begin{bmatrix} 2e^{-2t} - 3e^{-9t} \\ 4e^{-2t} + 9e^{-9t} \end{bmatrix}
\]

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