1. **RLC circuit**  
In this problem, we study the differential equations governing a series RLC circuit, which we solve to get the transient behavior. We consider the simple RLC circuit below. Suppose that the switch is closed at time $t = 0$.

![RLC Circuit Diagram](image)

(a) Write the voltages $v_C, v_R, v_L$ in terms of the current $i$, with respect to time $t$.

**Answer:**

\[ v_C = \frac{q}{C} = \frac{1}{C} \int_0^t i \, dt \]
\[ v_R = iR \]
\[ v_L = L \frac{di}{dt} \]

(b) Write down a second order differential equation for the current in the circuit with respect to time, in terms of the constants $R, L, C$.

**Answer:**

\[ \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{LC} i = 0 \]

(c) Rewrite your second order differential equation in the form

\[ \begin{pmatrix} \frac{di}{dt} \\ \frac{dv_L}{dt} \end{pmatrix} = A \begin{pmatrix} i \\ v_L \end{pmatrix} \]

where $A$ is a $2 \times 2$ matrix with coefficients that depend only on $R, L, C$.

**Answer:**

\[ \begin{pmatrix} \frac{di}{dt} \\ \frac{dv_L}{dt} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} i \\ v_L \end{pmatrix} \]
(d) Find the eigenvalues and corresponding eigenvectors of your matrix $A$ from the previous part.

**Answer:**

$$\lambda = \frac{R}{2L} \pm \sqrt{\left( \frac{R}{2L} \right)^2 - \frac{1}{LC}}$$

$$\vec{v} = \begin{pmatrix} 1 \\ L\lambda \end{pmatrix}$$

(e) For the case where the two eigenvalues are real, we claim that the solution to this system of differential equations is of the form

$$\begin{pmatrix} i \\ v_L \end{pmatrix} = c_0 e^{\lambda_0 t} \vec{v}_0 + c_1 e^{\lambda_1 t} \vec{v}_1,$$

where $c_0, c_1$ are constants, and $\lambda_0, \lambda_1$ are the eigenvalues of $A$ with eigenvectors $\vec{v}_0, \vec{v}_1$ respectively. Solve for the constants $c_0, c_1$, with the initial conditions $i = 0, v_L = 1$ at $t = 0$. Write your solution for $i$ as a function of $t$.

**Answer:**

$$\alpha = \frac{R}{2L}$$

$$\gamma = \sqrt{\left( \frac{R}{2L} \right)^2 - \frac{1}{LC}}$$

$$i = \frac{1}{2\gamma L} e^{-\alpha t} (e^{\gamma t} - e^{-\gamma t})$$

(f) For the case where the eigenvalues are complex, the solution to the system has the same form as in the previous part. Find $i$ as a function of $t$ in this case.

**Answer:**

$$\alpha = \frac{R}{2L}$$

$$\beta = \sqrt{\frac{1}{LC} - \left( \frac{R}{2L} \right)^2}$$

$$i = \frac{1}{2i\beta L} e^{-\alpha t} (e^{i\beta t} - e^{-i\beta t}) = \frac{1}{\beta L} e^{-\alpha t} \sin \beta t$$

2. **RLC circuit in AC**

We study a simple RLC circuit with an AC voltage source given by

$$v_s = B \cos(\omega t - \phi)$$

![RLC circuit diagram]
(a) Write out the phasor representation of $v_s, R, C, L$.

**Answer:** $V_s = Be^{-i\phi}$, $Z_R = R$, $Z_C = \frac{1}{j\omega C}$, $Z_L = j\omega L$.

(b) Use Kirchhoff’s laws to write down a loop equation relating the phasors in the previous part.

**Answer:**

$$Z_R I + Z_C I + Z_L I = V_s \tag{5}$$

$$\left(R + \frac{1}{j\omega C} + j\omega L\right)I = Be^{-i\phi} \tag{6}$$

(c) Solve the equation in the previous step for the current $I$. What is the polar form of $I$?

**Answer:**

$$|I| = \frac{B}{\sqrt{R^2 + \left(-\frac{1}{\omega C} + \omega L\right)^2}}$$

$$\angle I = -\phi - \tan^{-1}\left(\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right)\right)$$

(d) Compute the polar form of $V_R, V_L, V_C$.

**Answer:**

$$|V_R| = \frac{BR}{\sqrt{R^2 + \left(-\frac{1}{\omega C} + \omega L\right)^2}}$$

$$\angle V_R = \angle I = -\phi - \tan^{-1}\left(\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right)\right) = -\phi - \theta$$

$$|V_C| = \frac{B}{\sqrt{(1 - \omega^2 LC)^2 + (R\omega C)^2}}$$

$$\angle V_C = -\phi - \theta - \frac{\pi}{2}$$

$$|V_L| = \frac{\omega L B}{\sqrt{R^2 + \left(-\frac{1}{\omega C} + \omega L\right)^2}}$$

$$\angle V_L = -\phi - \theta + \frac{\pi}{2}$$

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