1. Linearization of functions

(a) Linearize $\sin(\theta)$ for $|\theta| < 5^\circ$

Answer: $\sin(x) \approx x$

(b) Linearize $\cos(\theta)$ for $|\theta| < 5^\circ$

Answer: $\cos(x) \approx 1$

(c) Linearize $\sqrt{x}$ for $0 \leq x < 0.1$

Answer:

$$\sqrt{x} \approx f(a) + \frac{1}{2\sqrt{a}}(x-a) \tag{1}$$

where $a = 0.05$.

2. Linear Temperature Control System

Imagine you are designing a room temperature controller. As a designer, you have the following tools: (1) a digital thermometer that gives you a temperature measurement in digital form and (2) a heat pump (air conditioner) which can warm up or cool down the room or do nothing.

Now you would like to keep the room temperature around $T^\circ F$.

(a) Describe how you will intuitively design the controller using English. Use a combination of English and math where $x$ is the temperature and $u$ is the setting on the heat pump (positive $u$ pumps heat into the room and negative $u$ pumps heat out of the room at the rate of $u$ Watt.)

Answer:

Our state is the current temperature in the room, $x$, and we want our controller to heat up when $x$ is below $T$ and to cool down when $x$ is above $T$.

(b) Now assume the heat pump and controller together set the $u$ so that the room temperatures evolve according to the following differential equation: $\frac{dx}{dt} = k_1(T - x)$, where $k_1 > 0$. Assume $x(0) = 0.8T^\circ F$, what is $x(t)$ for $t > 0$?

Answer:

Let $z = T - x$, we have $-\frac{dz}{dt} = k_1z$, whose solution is given by $z(t) = ce^{-k_1t}$. Using the initial condition, we have $c = 0.2T$. Therefore, $x(t) = T - z(t) = T - 0.2Te^{-k_1t}$.

(c) What was the feedback control law $u(t) = g(T, x(t))$ used to get the above behavior, assuming there is no heat loss or gain elsewhere? (Let $c$ be the heat capacity of the room. So $c$ Joules of added heat are required to raise the room temperature by $1^\circ F$)

Answer:

We should set $u(t)$ such that it gives rise to the rate of change for the temperature in the room, $\frac{dx}{dt} = k_1(T - x)$. This rate corresponds to $c\frac{dx}{dt}$ as the rate of Joules transferred to the room due to $u$. Therefore, we have $u(t) = ck_1(T - x)$.
(d) In the real world, the heat pump will not change its behavior instantaneously, and your temperature sensor will report the temperature once a minute. Let’s assume the heat pump is synchronized to the temperature reader and also modifies its behavior once a minute. We can write down a difference equation (discrete-time model) for the controlled air conditioner as \( x[t + 1] = x[t] + \Delta x = x[t] + k_2(T - x[t]) \). Assume \( x[0] = 0.8T^\circ F \), write down \( x[t] \) for \( t > 0 \).

**Answer:**

The difference equation is given by:

\[
x[t + 1] = x[t](1 - k_2) + k_2T
\]  

(2)

By change of variable, \( z[t] = T - x[t] \), we have:

\[
T - z[t + 1] = (T - z[t])(1 - k_2) + k_2T
\]  

(3)

\[
z[t + 1] = z[t](1 - k_2)
\]  

(4)

The solution is \( z[t] = z[0](1 - k_2)^t \). Since \( z[0] = 0.2T \), we have \( x[t] = T - 0.2T(1 - k_2)^t \).

(e) (optional, for now) What is the relationship between \( k_2 \) and \( k_1 \)?

**Answer:**

We have \( x[t + 1] = x[t] + \frac{60u[t]}{c^2} \), substituting our control strategy that \( u(t) = ck_1(T - x) \), we have:

\[
x[t + 1] = x[t] + 60k_1(T - x[t])
\]  

(5)

By comparison, we have \( k_2 = 60k_1 \).

(f) Consider the above difference equation, does bigger \( k_2 \) imply faster convergence? How about \( k_1 \) in the differential equation case?

**Answer:**

From the solution we have the following cases:

i. \( k_2 = 1 \): the temperature is achieved immediately.

ii. \( k_2 \in (1,2) \) or \( k_2 \in (0,1) \): oscillation to approach \( T \).

iii. \( k_2 = 2 \): only oscillation.

iv. \( k_2 > 2 \): oscillate away (out of control).

v. \( k_2 = 0 \): instantly go to \( 0.8T \).

vi. \( k_2 < 0 \): exponentially decreasing.

For \( k_1 \), as we have \( x(t) = T - 0.2T \exp^{-kt} \), we have

i. \( k_1 > 0 \): approaching \( T \)

ii. \( k_1 = 0 \): constantly \( 0.8T \)

iii. \( k_1 < 0 \): keep decreasing

3. **Linearizing a Nonlinear System**

Consider the following two-dimensional system. There are two states \( x_0 \) and \( x_1 \) and we can apply two inputs \( u_0 \) and \( u_1 \). The system evolves according to the following coupled differential equations:

\[
\frac{d}{dt}x_0(t) = x_0^2(t) + 2x_1(t) + 2u_0(t) + x_1(t)u_1(t)
\]  

(6)

\[
\frac{d}{dt}x_1(t) = x_1^2(t) + 2x_0(t) + u_1(t) + x_0(t)u_0(t)
\]  

(7)
(a) Write the above into the standard form \( \frac{d}{dt} \vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t)) \).

**Answer:**

\[
\frac{d}{dt} \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} = \begin{bmatrix} x_0^2(t) + 2x_1(t) + 2u_0(t) + x_1(t)u_1(t) \\ x_1^2(t) + 2x_0(t) + u_1(t) + x_0(t)u_0(t) \end{bmatrix}
\] (8)

(b) For the above system, assume that \( \vec{u} = 0 \) for all time. For what values of \( \vec{x} \) is \( \vec{f}(\vec{x}, \vec{0}) = \vec{0} \)? These are potential operating points where the control could be zero.

**Answer:** Plugging in \( \vec{f}(\vec{x}, \vec{0}) = \vec{0} \), we have

\[
x_0^2 + 2x_1 = 0 \quad (9)
\]
\[
x_1^2 + 2x_0 = 0 \quad (10)
\]

Solving it, we have

\[
x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad x = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \quad (11)
\]

(c) For the above system, linearize the dynamics around both of those potential operating points.

**Answer:** Let the operating point be \( \vec{x}_{op} \). The function \( \vec{f} \) evaluated at \( \vec{x}_{op} \) is \( \vec{f}(\vec{x}_{op}, \vec{0}) = \vec{0} \).

The linearization is

\[
\frac{d}{dt} \vec{x} = \vec{f}(\vec{x}_{op}) + \frac{\partial \vec{f}}{\partial \vec{x}} \bigg|_{(\vec{x}_{op}, \vec{0})} (\vec{x} - \vec{x}_{op}) + \frac{\partial \vec{f}}{\partial \vec{u}} \bigg|_{(\vec{x}_{op}, \vec{0})} \vec{0}
\] (12)

\[
= J_{\vec{x}}(\vec{x}_{op})(\vec{x} - \vec{x}_{op})
\] (13)

Let \( J_{\vec{x}} \) be \( \frac{\partial \vec{f}}{\partial \vec{x}} \). The matrix \( J_{\vec{x}} \) is the Jacobian matrix denoted by \((\vec{u} = \vec{0})\)

\[
J_{\vec{x}} = \begin{bmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_0}{\partial x_1} \\ \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 2x_0 & 2 \\ 2 & 2x_1 \end{bmatrix}
\] (14)

For \( x = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \) we have

\[
J_{\vec{x}} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}
\] (15)

For \( x = \begin{bmatrix} -2 & -2 \end{bmatrix}^T \), we have

\[
J_{\vec{x}} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}
\] (16)

(d) For the above linearized systems, what are the eigenvalues of the resulting \( A \) matrices for both of those operating points?

**Answer:** For \( x = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \) we have \( \lambda = \pm 2 \).

For \( x = \begin{bmatrix} -2 & -2 \end{bmatrix}^T \), we have \( \lambda = -2, -6 \).

(e) Can you linearize around \( \vec{x} = [-1, 0]^T \)? Is there a control \( \vec{u} \) that will keep it there?

**Answer:** Yes, we can linearize around any \( \vec{x} \).

Yes. Solving for \( \vec{f}(\vec{x}, \vec{u}) = 0 \), we have \( \vec{u} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}^T \).

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