This homework is due April 11, 2016, at Noon.

1. Homework process and study group

Who else did you work with on this homework? List names and student ID’s. (In case of hw party, you can also just describe the group.) How did you work on this homework?

**Solution:** I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.

2. Lecture Attendance

Did you attend live lecture this week? (the week you were working on this homework) What was your favorite part? Was anything unclear? Answer for each of the subparts below. If you only watched on YouTube, write that for partial credit.

(a) Monday lecture
(b) Wednesday lecture
(c) Friday lecture

**Solution:** Full credit for attending live lecture and giving a comment (what you liked best, what was unclear) about that lecture. 8 points for attending live lecture but giving no comment. 5 points for watching on YouTube and giving a comment. 2 points for just watching on YouTube. 0 points for blank or not watching lecture at all.

3. Redo problem 3 on the midterm

(a)
(b)
(c)
(d)
(e)
(f)
(g)

4. Redo problem 4 on the midterm

(a)
(b)
(c)
5. Redo problem 5 on the midterm
   (a)
   (b)

6. Redo problem 6 on the midterm
   (a)
   (b)
   (c)

7. Redo problem 7 on the midterm
   (a)
   (b)

8. Redo problem 8 on the midterm
   (a)
   (b)
   (c)
   (d)
   (e)
   (f)

9. Redo problem 9 on the midterm
   (a)
   (b)

10. Redo problem 10 on the midterm

11. Redo problem 11 on the midterm

12. Midterm comments
    Please give your feedback and comments on the midterm. Did you find it fair? What could we have done to have made this exam better?

13. SVD for minimum energy control
    Consider the open-loop discrete-time linear system with state $\vec{x}(t)$ and scalar control $u(t)$ defined by the recursive equation:
    \[ \vec{x}(t+1) = A\vec{x}(t) + Bu(t). \]
    We want to drive the system from some initial state $\vec{x}_0$ to $\vec{x}_f$. We know that if $A, B$ are controllable and the dimension is $n$, then clearly we can get to the desired $\vec{x}_f$ in $n$ steps. However, suppose that we only need to get there by $m > n$ steps. We now have a lot of flexibility. How can we choose a best control to apply?
    One choice is to ask for a control that gets us to the desired $\vec{x}_f$ using minimal energy. i.e., having minimal
    \[ \sum_{t=0}^{m-1} |u(t)|^2. \]
    This often can make sense when the underlying units of the $u(t)$ is an applied voltage or some other voltage-like quantity because then the energy consumed is proportional to the square of $u(t)$.
(a) Consider the system evolution equations from $t = 1$ to $t = m$, obtain an expression for the final state $\vec{x}(m)$ as a function of the initial state $\vec{x}_0$ and control inputs.

Solution: By iteratively representing the state of $\vec{x}$ using the system equation, we have

$$\vec{x}(1) = A\vec{x}(0) + Bu(0)$$
$$\vec{x}(2) = A\vec{x}(1) + Bu(1) = A(A\vec{x}(0) + Bu(0)) + Bu(1) = A^2\vec{x}(1) + ABu(0) + Bu(1)$$
$$\vdots$$
$$\vec{x}(m) = A^m\vec{x}(0) + A^{m-1}Bu(0) + A^{m-2}Bu(1) + \cdots + ABu(m-2) + Bu(m-1)$$

where we have obtained an expression for the final state $\vec{x}(m)$ as a function of the initial state $\vec{x}_0$ and control inputs.

(b) Write out the above equation in a giant matrix form, with $\vec{u} = [u(0), u(1), \ldots, u(m-1)]^T$.

Solution: From the previous part, we see that $\vec{x}(m)$ is the sum of $A^m\vec{x}(0)$ and $u(t)$’s that scale vectors of the form $A^iB$. We can write this in terms of matrix-vector multiplication as follows:

$$\vec{x}(m) - A^m\vec{x}(0) = [A^{m-1}B \ A^{m-2}B \ \cdots \ AB \ B] \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(m-2) \\ u(m-1) \end{bmatrix}$$

because the $A^m\vec{x}(0)$ term is not part of the giant matrix form, we move that to the left of the equation.

(c) Now you have obtained a linear equation in the form $\vec{y} = G\vec{u}$, where $\vec{y}$ and $G$ were found in the previous parts. Recall that in HW4 problem 3, you have shown that the solution obtained by means of the psuedo-inverse (computed using the SVD) has a nice minimality property. Use this to derive the minimum energy control inputs $\vec{u}$.

Solution: We know from HW4 problem 3 last part, that if $\vec{u}$ is the psuedo-inverse solution to $\vec{y} = G\vec{u}$, then $\|\vec{u}\| \leq \|\vec{y}\|$ for all other vectors $\vec{z}$ satisfying $\vec{y} = C\vec{z}$.

Now in order to obtain minimum energy control inputs, observe that it is equivalent to finding the control inputs having minimum norm $\|\vec{u}\|$. Hence all we have to do is to find the psuedo-inverse solution to $\vec{y} = G\vec{u}$, by using SVD as was discussed in HW4 problem 3. Specifically, $G^\dagger = V\Sigma^\dagger U^T$, where $V$, $\Sigma$, and $U$ come from the SVD of $G$, and $\Sigma^\dagger$ is a matrix with the reciprocal of all nonzero entries in the diagonal matrix $\Sigma$. We then have $\vec{u} = G^\dagger\vec{y}$.

The matrix $\Sigma$ is a diagonal matrix with the singular values $\sigma_i$ placed on the diagonal entries in descending order. This matrix is inverted as follows:

$$\Sigma^\dagger = \begin{bmatrix} 1/\sigma_0 \\ 1/\sigma_1 \\ \cdots \\ 1/\sigma_n \end{bmatrix}$$

If some $\sigma_i = 0$, then the corresponding entry in $\Sigma^\dagger$ is 0 instead of $1/\sigma_i$.

(d) How could you extend the above to the case when $u(t)$ is a vector $\vec{u}(t)$?
**Solution**: Put them in a long vector by stacking the \( \vec{u}(t) \) on top of each other. If \( \vec{u}(t) = \begin{bmatrix} u_0(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} \) is a 3-dimensional vector, for example, we would form the giant vector as follows:

\[
\begin{bmatrix}
  u_0(0) \\
  u_1(0) \\
  u_2(0) \\
  u_0(1) \\
  u_1(1) \\
  u_2(1) \\
  \vdots
\end{bmatrix}
\]

We then use exactly the same argument by constructing the giant matrix form from part (b).

(e) Let’s revisit Beiber’s segway from [Homework 3](#) from EE16A. In that problem, we had a 4-dimensional state vector

\[
\vec{x} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix}
\]

We modeled the motion of the segway as a cart-pole system, which was linearized to be a linear system. We are given the following information:

\[
A = \begin{bmatrix}
  1 & 0.05 & -0.01 & 0 \\
  0 & 0.22 & -0.17 & -0.01 \\
  0 & 0.10 & 1.14 & 0.10 \\
  0 & 1.66 & 2.85 & 1.14 \\
\end{bmatrix}
\]

\[
\vec{b} = \begin{bmatrix}
  0.01 \\
  0.21 \\
  -0.03 \\
  -0.44 \\
\end{bmatrix}
\]

\[
\vec{x}[0] = \begin{bmatrix}
  -0.3853493 \\
  6103227 \\
  0.9120005 \\
  -24
\end{bmatrix}
\]

Previously, we wanted to find the controls needed to reach the target state \( \vec{x}_f = \vec{0} \) in exactly 4 time steps. Because we chose 4 steps, the problem had exactly one solution. However, if we now increase the number of steps to \( m > 4 \), then there are infinitely many solutions to this problem. Let’s use what we learned from the previous parts of this question to find the minimum energy solution \( \sum_{t=0}^{m-1} ||u(t)||^2 \) when we set \( m = 30 \). Fill in the missing code in the IPython notebook to compute the minimum energy solution to this problem.

**Solution**: See [IPython notebook](#)
14. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

\[
\vec{x}(t+1) = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t).
\]

In standard language, we have \( A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \), \( B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) in the form: \( \vec{x}(t+1) = A\vec{x}(t) + Bu(t) \).

(a) Is this system controllable?

**Solution:** We calculate

\[
C = [B, AB] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

Observe that \( C \) matrix is full rank and hence our system is controllable.

(b) Is this discrete-time linear system stable on its own?

**Solution:** We have to calculate the eigenvalues of matrix \( A \). Thus,

\[
\det(\lambda I - A) = 0
\]

\[
\det \begin{bmatrix} \lambda + 2 & -2 \\ 2 & \lambda - 3 \end{bmatrix} = 0
\]

\[
\lambda^2 - \lambda - 2 = 0
\]

\[
\lambda_1 = 2, \lambda_2 = -1
\]

Thus, magnitude of eigenvalue \( \lambda_1 \) is greater than 1 and the system is unstable.

(c) Suppose we use state feedback of the form \( u(t) = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}(t) \)

Find the appropriate state feedback constants, \( f_1, f_2 \) so that the state space representation of the resulting closed-loop system has eigenvalues at \( \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2} \).

**Solution:** The closed loop system using state feedback has the form

\[
\vec{x}(t+1) = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot (f_1 \ f_2) \vec{x}(t) = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot [f_1 \ f_2] \vec{x}(t)
\]

Thus, the closed loop system has the form

\[
\vec{x}(t+1) = \begin{bmatrix} -2 + f_1 & 2 + f_2 \\ -2 + f_1 & 3 + f_2 \end{bmatrix} \vec{x}(t)
\]

Thus, finding the eigenvalues of the above system we have

\[
\det(\lambda I - \begin{bmatrix} -2 + f_1 & 2 + f_2 \\ -2 + f_1 & 3 + f_2 \end{bmatrix}) = 0 \Rightarrow \lambda^2 - (1 + f_1 + f_2)\lambda + f_1 - 2 = 0
\]

However, we want to place the eigenvalue at \( \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2} \). Thus, this means that
\[
\lambda^2 - (1 + f_1 + f_2) \lambda + f_1 - 2 = (\lambda + \frac{1}{2})(\lambda - \frac{1}{2})
\]

\[
\lambda^2 - (1 + f_1 + f_2) \lambda + f_1 - 2 = \lambda^2 - \frac{1}{4}
\]

Equating the co-efficients of the different powers of \(\lambda\) on both sides of the equation, we get,

\[
1 + f_1 + f_2 = 0
\]

\[
f_1 - 2 = -\frac{1}{4}
\]

The above system of equations gives us \(f_1 = \frac{7}{4}, f_2 = -\frac{11}{4}\)

(d) Now suppose we’ve got a seemingly different system described by the controlled scalar difference equation \(z(t + 1) = z(t) + 2z(t - 1) + u(t)\). (Think back to the Fibonacci number problem you saw in 16A.) Write down the above system’s representation in the following matrix form:

\[
\vec{z}(t + 1) = A_1 \vec{z}(t) + B_1 u(t).
\]

Please specify what the vector \(\vec{z}(t)\) consists of as well as the matrix \(A_1\) and the vector \(B_1\).

**Solution:** From the problem, we have \(z(t + 1) = z(t) + 2z(t - 1) + u(t)\). Define our state variable as \(\vec{z}(t) = \begin{bmatrix} z(t) \\ z(t + 1) \end{bmatrix}\), we can write the equation equivalently into the matrix form,

\[
\begin{bmatrix} z(t) \\ z(t + 1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} z(t - 1) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
\]

where \(A_1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\), \(B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\). It can be seen to be in canonical controllable form.

(e) We will now show how the initial matrix representation for \(\vec{x}(t)\) can be converted to the canonical form for \(\vec{z}(t)\) using a change of basis. Suppose we do a transformation of the coordinates of the state \(\vec{x}(t)\) to \(\vec{z}(t) = P\vec{x}(t)\). Write down the state-transition matrices of \(\vec{z}(t)\) in terms of the state transition matrices of \(\vec{x}(t)\), i.e., express \(A_1\) and \(B_1\) in terms of \(A, B, \) and \(P\).

**Solution:** Given that the new vector is transformed by the following matrix, \(\vec{z}(t) = P\vec{x}(t)\). As we know from before, \(\vec{x}(t + 1) = A\vec{x}(t) + Bu(t)\).

Now,

\[
\vec{z}(t + 1) = P\vec{x}(t + 1)
\]

\[
\vec{z}(t + 1) = P(A\vec{x}(t) + Bu(t))
\]

\[
\vec{z}(t + 1) = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \vec{z}(t) + \begin{bmatrix} PB \\ 0 \end{bmatrix} u(t)
\]

Thus,

\[
A_1 = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} A \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}^{-1}
\]

\[
B_1 = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} B
\]
(f) For $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, confirm that the state space representation of $\vec{z}(t)$ is the same as part (d). Design a feedback $\begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix}$ to place the closed-loop eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Confirm that $\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} P$.

**Solution:** We confirm that,

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = PAP^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

Also, confirm that

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = PB = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving for the new feedback matrix: The closed loop system using state feedback has the form

$$\vec{z}[t+1] = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \left( \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} \vec{z}(t) \right) = \left( \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} \right) \vec{z}(t)$$

Thus, the closed loop system has the form

$$\vec{z}(t+1) = \begin{bmatrix} 0 & 1 \\ 2 + \bar{f}_1 & 1 + \bar{f}_2 \end{bmatrix} \vec{z}(t)$$

Thus, finding the eigenvalues of the above system we have

$$\det(\lambda I - \begin{bmatrix} 0 & 1 \\ 2 + \bar{f}_1 & 1 + \bar{f}_2 \end{bmatrix}) = 0 \Rightarrow \lambda^2 - (1 + \bar{f}_2)\lambda - (2 + \bar{f}_1) = 0$$

However, we want to place the eigenvalue at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Thus, this means that

$$\lambda^2 - (1 + \bar{f}_2)\lambda - \bar{f}_1 - 2 = (\lambda + \frac{1}{2})(\lambda - \frac{1}{2})$$

$$\lambda^2 - (1 + \bar{f}_2)\lambda - \bar{f}_1 - 2 = \lambda^2 - \frac{1}{4}$$

Equating the co-efficients of $\lambda$ on both sides, we get

$$1 + \bar{f}_2 = 0$$
$$-\bar{f}_1 - 2 = -\frac{1}{4}$$

The above system of equations gives us $\bar{f}_1 = \frac{7}{4}, \bar{f}_2 = -1$

Matrix multiplication shows that

$$\begin{bmatrix} \frac{7}{4} & -\frac{1}{4} \\ \frac{1}{4} & -1 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} P = \begin{bmatrix} -\frac{7}{4} & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$
(g) Here, we gave you the $P$ matrix. How would you have come up with the $P$ matrix on your own? (Hint: start with the second column of $P$ and ask where it might have come from. Then, is there a relationship between the coefficients of the difference equation in part (d) to the polynomial whose roots you need to find in part (b)?)

By following this technique, any controllable discrete-time system can be converted to the “controllable canonical form” shown in part (d) by finding the right change of coordinates.

**Solution:**

The controllability matrix for the original system is given by

$$G = \begin{bmatrix} B & AB \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

As discussed in lecture, $G^{-1}$ is the basis which transforms the system to the transpose of the controllable canonical form ($\tilde{A}, \tilde{B}$).

$$\tilde{A} = G^{-1}AG = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

The target matrix is therefore the transpose of $\tilde{A}$ and is given by $A_1 = \tilde{A}^T = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The controllability matrix of the controllable canonical system is given by

$$H = \begin{bmatrix} B_1 & A_1B_1 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Again as discussed in lecture, $H^{-1}$ is the basis which transforms the system in controllable canonical form to the transpose of the controllable canonical form.

We can confirm this by computing,

$$\tilde{A} = H^{-1}A_1H = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore,

$$A_1 = H\tilde{A}H^{-1} = HG^{-1}AGH^{-1}$$

$$P = HG^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

And thus it all works and we have found the $P$ matrix from scratch. There are other ways that work as well. For example, we can find the basis $P^{-1}$ first by placing $B$ as the last column of it and then seeing that the controllable canonical form insists that $AB = 1\tilde{p}_0 + B$ where $\tilde{p}_0$ is the first basis element, or first column of $P^{-1}$. Consequently, $\tilde{p}_0 = AB - B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. At that point, $P$ can be found by inverting the matrix $[\tilde{p}_0B]$. Once the characteristic polynomial has been found, this trick always works (you need to use the coefficients of the characteristic polynomial as weights in the algorithm) and lets you recursively calculate the basis elements that will give you controllable canonical form.
However, although algorithmically simple, it is not obvious that it must work because this algorithm never explicitly relies on the controllability of $A, B$. The advantage of the recipe given in class is that it doubles as a valid proof.

(h) We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrix $A$ and $B$ from (a) for the system

$$\tilde{x}(t+1) = A\tilde{x}(t) + Bu(t) + w(t)$$

into the IPython notebook and use the random input $w(t)$ as the disturbance introduced into the state equation. Observe how the norm of $\tilde{x}(t)$ evolves over time for the given $A$. What do you see happening to the norm of the state?

**Solution:** See IPython notebook for solution. The norm of $\tilde{x}(t)$ increases with time for the given $A$. This is because the matrix $A$ has eigenvalues with magnitude greater than one as we discussed in (b) and thus the state keeps growing at each time step.

(i) Add the feedback computed in part (c) to the system in the notebook and explain how the norm of the state changes.

**Solution:** The eigenvalues of the closed loop system are at $\frac{1}{2}$ and $-\frac{1}{2}$. Thus, the norm of the state variable is now bounded with time. Check the solution in the IPython notebook.

(j) Now we evaluate a system described by the following scalar system $z(t+1) = az(t) + u(t) + w(t)$ in the iPython notebook. Consider two values of $a$, one case with $a > 1$ and one with $a < 1$, to observe the difference in the evolution of $|z(t)|$ for the same error as part (g). Describe the differences between the two.

**Solution:** For $|a| > 1$, the norm of $\tilde{z}(t)$ grows with time and is not bounded, while it is bounded for $|a| < 1$. This is because the eigenvalue of the evolution is given by `$a$' itself and it determines if the state is stable or not depending on its magnitude. Check the solution in the IPython notebook.

(k) Suppose that the disturbance is actually coming from observation noise. We assume $y(t) = x(t) + w(t)$ where $w(t)$ is some random noise. Add a state feedback $u(t) = ky(t)$ to the system so that the resulting closed loop system is described by $z(t+1) = (a+k)z(t) + kw(t)$. Say we know $a = -3$. For what values of $k$’s will the result be bounded. Confirm with the norm of the closed loop system.

**Solution:** The system will be stable for $|a+k| < 1$. For these values of $k$, the state will not grow exponentially. Check the solution on the IPython notebook. Therefore, $k > 2$ and $k < 4$.

(l) Is it advisable to have $a+k$ close to zero if we want to minimize the magnitudes of the state $x(t)$? How does the effect of the noise in the observation influence this? Assign values of $k$ close to $-a$ to see the effect in the IPython notebook.

**Solution:** No. It is not advisable to have $a+k$ close to zero. This is because the effect of error $w(t)$ which is weighted by $k$ will keep increasing for large values of $k$ and a spurious error will have significant effect on the state even though the state is BIBO stable. The effect can be seen in the IPython notebook for $k = 2.99$ and $k = 2.8$, where the instantaneous value of $z(t)$ can go to larger values.

15. Your Own Problem

Write your own problem related to this week’s material and solve it. You may still work in groups to brainstorm problems, but each student must submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

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