This homework is due March 28, 2016, at Noon.

1. Homework process and study group

Who else did you work with on this homework? List names and student ID’s. (In case of hw party, you can also just describe the group.) How did you work on this homework?

Solution: I worked on this homework with...
I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...
Then I went to homework party for a few hours, where I finished the homework.

2. Lecture Attendance

This question is a student trust-based system for giving credit to those who attend lecture and help the course out by doing so. Lying on this (or any other part of the homework) constitutes academic dishonesty, and more importantly than any academic sanctions, lying would damage your honor and integrity. Be honest. You will carry your honor and integrity with you for the rest of your life, and they are way more important than your GPA.

Did you attend live lecture this week? (the week you were working on this homework) What was your favorite part? Was anything unclear? Answer for each of the subparts below. If you only watched on YouTube, write that for partial credit.

(a) Monday lecture
(b) Wednesday lecture
(c) Friday lecture

Solution: Full credit for attending live lecture and giving a comment (what you liked best, what was unclear) about that lecture. 8 points for attending live lecture but giving no comment. 5 points for watching on YouTube and giving a comment. 2 points for just watching on YouTube. 0 points for blank or not watching lecture at all.

3. Solving Second-Order Differential Equations

In the lecture, you were shown how to solve first-order differential equations. We will ask you to solve the following equation, which involves second-order differentiations:

\[ \frac{d^2}{dt^2} y(t) - 2 \frac{d}{dt} y(t) - 3y(t) = 0 \]

where the initial conditions are \( y(0) = 0 \), \( \frac{dy}{dt}(0) = 1 \).

(a) Firstly, please write the above equations into the matrix form \( \frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \). How do you define \( \vec{x}(t) \), \( A \), and the initial conditions \( \vec{x}(0) \)?

Solution:
We can define the vector variable \( \vec{x}(t) = \begin{bmatrix} y(t) \\ \frac{d}{dt}y(t) \end{bmatrix} \), because then if we take the first derivative of \( \vec{x} \) we can obtain second derivatives. The second order differential equation can be written as:

\[
\frac{d}{dt} \vec{x}(t) = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \vec{x}(t) \tag{1}
\]

where the first row of matrix \( A \) is given by definition, and the second row is exactly given by the second order differential equation.

(b) What are the eigenvalues and eigenvectors of \( A \)?

**Solution:**

First, we write out the characteristic equation:

\[
det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 3 & 2 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0 \tag{2}
\]

Therefore, the eigenvalues are \( \lambda_0 = 3 \), \( \lambda_1 = -1 \). Now let’s find the eigenvectors for each case. For \( \lambda_0 = 3 \), define the vector \( \begin{bmatrix} u \\ v \end{bmatrix} \),

\[
\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ 3u + 2v \end{bmatrix} = 3 \begin{bmatrix} u \\ v \end{bmatrix} \tag{3}
\]

Therefore we have \( v = 3u \), in other words, \( \vec{v}_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) is the corresponding eigenvector. Similarly, for \( \lambda_1 = -1 \), we have:

\[
\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ 3u + 2v \end{bmatrix} = -\begin{bmatrix} u \\ v \end{bmatrix} \tag{4}
\]

Therefore we have \( v = -u \), in other words, \( \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) is the corresponding eigenvector.

(c) Based on the above results, what is the solution to the second-order equation?

**Solution:**

The general solution to the first order differential equation in the matrix form, as discussed in the lecture, is:

\[
\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{5}
\]

where \( c_1 \) and \( c_2 \) are constants that need to be determined. Using the initial conditions that \( \vec{x}(0) = \begin{bmatrix} y(0) \\ \frac{d}{dt}y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), we have

\[
c_1 + c_2 = 0 \\
-c_1 + 3c_2 = 1
\]

where we know that \( c_1 = -\frac{1}{4}, c_2 = \frac{1}{4} \). Therefore, we have \( \vec{x}(t) = -\frac{1}{4} e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{4} e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \). From the above, we can see that the solution for the second order differential equation is:

\[
y(t) = -\frac{1}{4} e^{-t} + \frac{1}{4} e^{3t}. \tag{6}
\]
Figure 1: A spring model with one end of the spring attached to a point \( \vec{c} \) and the other end attached to some particle with mass \( m = 1 \) at point \( \vec{x} \). The rest length of the spring is \( L \) and is denoted by spring on the bottom. The spring on the top has been stretched past the rest length, and therefore, the acceleration \( \vec{a} \) is pointing towards the left.

4. Solving Differential Equations Numerically

In this problem, we will be looking at how you can use a computer to solve differential equations. Sometimes differential equations cannot be solved analytically, and when this happens, we can use numerical algorithms to find a close approximation to the answer. In this problem, we will consider the motion over time of a particle attached to a spring. We set the mass of the particle to 1 so that we can compute the acceleration from the force \( \vec{F} \) as \( \vec{F} = \vec{a} \). See Figure 1 for a visualization of the spring model. We model the motion of a particle by its position \( \vec{x}(t) \), its velocity \( \vec{v}(t) \), and its acceleration \( \vec{a}(t) \), and these three functions are related by

\[
\frac{d\vec{x}(t)}{dt} = \vec{v}(t) \quad \text{and} \quad \frac{d\vec{v}(t)}{dt} = \vec{a}(t).
\]

If we are modeling a particle with mass 1, the acceleration is given by Hooke’s Law:

\[
\vec{a}(t) = - (k_s (||\vec{x}(t) - \vec{c}|| - L)) \frac{\vec{x}(t) - \vec{c}}{||\vec{x}(t) - \vec{c}||} \quad (7)
\]

where \( k_s \) is the spring constant, \( \vec{c} \) is the location of the other end of the spring, and \( L \) is the rest length of the spring. Note that all of these functions \( \vec{x}(t) \), \( \vec{v}(t) \), and \( \vec{a}(t) \) are continuous functions that we can evaluate at any time \( t \).

(a) In this problem, we will use a computer to approximate the solution to the differential equation. Due to limitations of the computer, we will take our continuous function \( \vec{x}(t) \) and solve for it at discrete points in time \( \vec{x}(t_0), \vec{x}(t_1), \ldots, \vec{x}(t_n) \). Given the acceleration of a particle \( \vec{a}(t) \), we will use the Euler method to approximate the \( \vec{x}(t) \) at these discrete points in time. However, we will still use our knowledge of the continuous functions \( \vec{v}(t) \) and \( \vec{a}(t) \) to help us make the approximation. Euler’s method works...
with first-order differential equations, but because we have an equation for acceleration, our problem is second-order. To work around this problem, we define a new function

$$\vec{y}(t) = \left( \frac{\vec{x}(t)}{\vec{v}(t)} \right)$$

Please write the derivative \( \frac{d\vec{y}(t)}{dt} \) in terms of \( \vec{x}(t) \) and \( \vec{v}(t) \).

**Solution:** We begin by expressing the derivative of \( y(t) \) by its components:

$$\frac{d\vec{y}(t)}{dt} = \left( \frac{d\vec{x}(t)}{dt}, \frac{d\vec{v}(t)}{dt} \right)$$

Using the fact that \( \frac{d\vec{x}(t)}{dt} = \vec{v}(t) \) and \( \frac{d\vec{v}(t)}{dt} = \vec{a}(t) \) we can replace the derivatives with \( \vec{v}(t) \) and \( \vec{a}(t) \) to obtain.

$$\frac{d\vec{y}(t)}{dt} = \left( \vec{v}(t), \vec{a}(t) \right)$$

Next, we use Equation 7 to write \( \vec{a}(t) \) in terms of \( \vec{x}(t) \). With this substitution, our final equation is

$$\frac{d\vec{y}(t)}{dt} = \left( \vec{v}(t), \vec{a}(t) \right)$$

(b) As you can see from the previous part, we now can write the derivative of \( \vec{y}(t) \) in terms of only parts of \( \vec{y}(t) \). Thus we can express the derivative as \( \frac{d\vec{y}(t)}{dt} = \vec{f}(t, \vec{y}(t)) \) where \( \vec{f} \) is the function you derived to compute the derivative in part (a). In this form, we can now use Euler’s method to approximate the function at time \( t + h \) by

$$\vec{y}(t + h) = \vec{y}(t) + h\vec{f}(t, \vec{y}(t))$$

where \( h \) is the length between each discrete point in time we are estimating \( \vec{y}(t) \).

Let’s test this method on a 1-dimensional spring example where one end of the spring is fixed and a particle is on the other end. In the supplied IPython Notebook, write the code to compute the acceleration of the particle in the function \( \text{spring}_d \) by writing the code to calculate \( \frac{d\vec{y}(t)}{dt} \).

**Solution:** See the IPython Notebook for the solution

(c) When you run Euler’s method and view the particle moving over time, you will notice that something appears wrong after some time. Describe what you see.

**Solution:** After each oscillation, the particle on the spring moves farther out, which add energy to the system. Because a spring model doesn’t add energy to the system, we know that this numerical method is adding some error to our solution at each step, which becomes visible after some time.

(d) To fix this problem, a common method is to add a “drag” force to the particle, which applies a force opposing the velocity of the particle. Adding this term to the acceleration, we now have

$$\vec{a}(t) = - (k_s (\||\vec{x}(t) - \vec{c}|| - L)) \frac{\vec{x}(t) - \vec{c}}{||\vec{x}(t) - \vec{c}||} - k_d\vec{v}(t)$$

where \( k_d \) is the drag constant that controls how strong the drag force is. In the IPython Notebook, write the computation for this new acceleration including the drag force in the function \( \text{spring}_\text{drag}_d \).

**Solution:** See the IPython Notebook for the solution
Although in the previous parts we have assumed that one end of the spring is fixed, the other end can also be attached to another particle. Let’s suppose we have two particles $A$ and $B$ whose positions are given by $\vec{x}_A(t)$ and $\vec{x}_B(t)$. Now, the acceleration of the two particles depends on the positions of both of them. From physics, we know that the force due to the spring on particle $B$ must be equal in magnitude and opposite in direction from the force on $A$. Using this information, write an expression for the accelerations $\vec{a}_A(t)$ and $\vec{a}_B(t)$ by using Equation 13 as a guide. The accelerations should include just the acceleration from the spring and not the drag.

**Solution:** As in the previous parts, we still use Equation 7 to find the acceleration of the particles. Because the two particles are attached to the ends of the spring, we can compute the length of the spring by the distance between the two particles: $\| \vec{x}_A(t) - \vec{x}_B(t) \|$, where $\vec{x}_A$ and $\vec{x}_B(t)$ are the positions of particles $A$ and $B$ at time $t$. Thus, the only changes we need to make in Equation 7 are to replace $\vec{x}(t) - \vec{c}$ with $\vec{x}_A(t) - \vec{x}_B(t)$ to find the acceleration on particle $A$. The equation is given by

$$\vec{a}_A(t) = -(k_s (\| \vec{x}_A(t) - \vec{x}_B(t) \| - L)) \frac{\vec{x}_A(t) - \vec{x}_B(t)}{\| \vec{x}_A(t) - \vec{x}_B(t) \|} \tag{14}$$

To compute the acceleration of particle $B$, we use the fact that the acceleration is equal in magnitude to $\vec{a}_A(t)$ and opposite in direction. Therefore, we negate $\vec{a}_A(t)$ to find the acceleration:

$$\vec{a}_B(t) = -\vec{a}_A(t) \tag{15}$$

Alternatively, we can compute the acceleration on particle $B$ with Equation 7 as well. This is given by

$$\vec{a}_B(t) = -(k_s (\| \vec{x}_B(t) - \vec{x}_A(t) \| - L)) \frac{\vec{x}_B(t) - \vec{x}_A(t)}{\| \vec{x}_B(t) - \vec{x}_A(t) \|} \tag{16}$$

Now that we have an equation to compute the acceleration from a spring between two pairs of particles, we can compute accelerations for a system of particles by computing the pair-wise spring accelerations. In the IPython Notebook write your equation in the function `spring_pair_d`. Run the animation at the end of the notebook to verify that you are computing the accelerations correctly.

**Solution:** See the IPython Notebook for the solution

5. Simulated Inductor - Gyrator

As you have seen in other parts of the homework, you can make really interesting and useful circuits using a combination of resistors, capacitors, and inductors. Yet, the inductance needed for certain uses require very large inductors who may not be feasible to put on a small chip.

One way to work around this constraint is to simulate an inductor using a circuit model called Gyrator. Based on the model, we can simulate an inductor using only a capacitor, two resistors, and one op-amp.
The RL Equivalent circuit is as follows:

\[ Z_{in} = R_L + j \omega R_C R_L C \]  

(17)

The 2nd figure shows that the input impedance for both circuits is defined as

\[ Z_{in} = R_L + j \omega R_C R_L C \]

Let us work through the steps to see that these two circuits are equivalent by showing that their impedances are very similar. In addition, we will show some cases where the Gyrator fails.

Assume that \( R_C \gg R_L \), \( C \) is very small, and \( V_{in} \) can only oscillate at small frequencies, unless otherwise stated.

Recall that the impedance \( Z_{in} \) into a circuit is defined as \( \frac{V_{in}}{I_{in}} = Z_{in} \).

Show all your work to receive credits. All answers should be in Phasor form.

(a) We can determine the total input impedance by solving each branch separately. First, determine the impedance \( Z_1 \), which is the impedance of the lower branch.

**Solution:** By definition of an Op-amp, the input impedance into the + terminal is infinite so no current will flow through there. As such, \( I_1 = \frac{V_{in}}{j \omega C + R_C} \) and we get

\[ Z_1 = \frac{V_{in}}{I_1} = R_C + \frac{1}{j \omega C} \]
(b) Determine the impedance $Z_2$. Hint: Use the properties of an op-amp you have learned.

**Solution:** To solve this problem we need to find the current $I_2$. As such, we get the voltage at the node $V_-$. By the rules of op-amps with feedback, $V_- = V_+$. We can get $V_+$ by seeing that the lower branch is a voltage divider. Hence,

$$V_+ = V_- = \frac{V_{\text{in}} \times R_C}{Z_1}$$

Now we can solve for the current $I_2$ because we can calculate the voltage drop across $R_L$. As such,

$$I_2 = \frac{\Delta V}{R_L} = (V_{\text{in}} - V_-) \times \frac{1}{R_L} = (V_{\text{in}} - \frac{V_{\text{in}} \times R_C}{Z_1}) \times \frac{1}{R_L} = V_{\text{in}} \times \frac{Z_1 - R_C}{Z_1 \times R_L}$$

Now that we know $I_2$, we get that

$$Z_2 = \frac{V_{\text{in}}}{I_2} = \frac{Z_1 \times R_L}{Z_1 - R_C} = (R_C + \frac{1}{j\omega C}) \times j\omega CR_L = R_L + j\omega CR_L R_C$$

Hence,

$$Z_2 = R_L + j\omega CR_L R_C$$

(c) Now that you have found the two parallel impedances, determine $Z_{\text{in}}$. Remember our assumptions and the properties of parallel impedance. Your final answer should be equation 17. Hint: The solution will rely on getting an approximation which will work under our assumptions.

**Solution:** Now that we have found the impedances of two branches, we note that they are parallel paths from $V_{\text{in}}$ to Ground. As such,

$$Z_{\text{in}} = Z_1 || Z_2$$

By our assumptions, we see that $Z_1 >> Z_2$. To see this, you can take the ratio $\frac{Z_1}{Z_2}$.

$$\frac{Z_1}{Z_2} = \frac{R_C + \frac{1}{j\omega C}}{R_L + j\omega CR_L R_C} = \frac{1}{R_L \times \left(1 + j\omega CR_C \right)} = \frac{1}{j\omega CR_L}$$

Since $\omega$, $C$, and $R_L$ are small then this ratio is large. Hence, we can state that $Z_1 >> Z_2$.

Another way to see this without making $R_L$ small is that we are making this circuit with a desired inductance $L$. As such, we want $C \times R_L \times R_C = L$ which gives us 3 degrees of freedom to arrive at our desired inductance. Now we can write the two impedances as

$$Z_1 = R_C + \frac{1}{j\omega C}$$
\[ Z_2 = R_L + j\omega L \]

Now we see that \( Z_2 \) has no dependence on \( R_C \) or \( C \). As such, we can make \( R_C \) sufficiently large and \( C \) sufficiently small such that \( Z_1 >> Z_2 \). And we can get our desired impedance because \( R_L \) is a free variable to determine \( L \).

Now that we know \( Z_1 >> Z_2 \), we apply what we know about parallel impedances in that the smaller impedance dominates. As such,

\[ Z_{in} = \frac{Z_1}{Z_2} = Z_2 = R_L + j\omega CR_L R_C \]

(d) Now, let’s check if the two circuits are equivalent under certain conditions. Let \( V_{in} \) be a DC voltage, i.e. its frequency is 0. Are the input impedances for the two circuits equivalent? If not, what is the input impedance for each circuit?

**Solution:** Input impedance are the same.

One can see this because this scenario falls under our assumptions that we made during the derivation in that \( V_{in} \) oscillates at very small frequencies. As such, the equivalence would still hold.

(e) Let \( V_{in} \) be an oscillating cosine with an infinitely large frequency. Are the input impedances for the two circuits equivalent? If not, what is the input impedance for each circuit?

**Solution:** Impedances are different.

Since the frequency is very high, the capacitor acts as a short. As such, \( V_+ = V_- = V_{in} \) so no current flows through \( R_L \). All the current will flow through \( R_C \).

Hence, the impedance for the first circuit would be \( R_C \). The impedance for the second circuit would be \( \infty \).

6. RLC Circuit 1

In this question, we will take a look at an electrical systems described by second order differential equations and analyze their transfer function. Consider the circuit below:

![RLC Circuit Diagram](image)

(a) Write the KVL equation for the above circuit in time domain and convert it to the phasor domain.

**Solution:** Suppose the current flowing through the circuit is \( i \). The voltage across the various components are given in the figure. The impedance of the capacitor is \( \frac{1}{j\omega C} \), while the impedance of the inductor is \( j\omega L \). Therefore, \( V_C = \frac{i}{j\omega C} \) and \( V_L = j\omega Li \). Applying KVL along the loop of the series circuit we get,

\[ V_s - V_C - V_R - V_L = 0 \]  (18)

\[ V_s = \frac{i}{j\omega C} + j\omega Li \]  (19)
Writing the voltages in terms of the current through the components in phasor domain,

\[
V_s - \frac{i}{j\omega C} - iR - ij\omega L = 0 \quad (20)
\]

\[
i = \frac{V_s}{j\omega C + R + j\omega L} \quad (21)
\]

(b) Suppose the voltage source was shorted at \( t \geq 0 \), i.e., \( V_C = V_s = V \) for \( t < 0 \) and \( V_s = 0 \) at \( t \geq 0 \). Write down the differential equations in the form of the matrix

\[
\begin{pmatrix}
\frac{di}{dt} \\
\frac{dV_C}{dt}
\end{pmatrix} = A
\begin{pmatrix}
i \\
V_C
\end{pmatrix} \quad (23)
\]

**Solution:** Knowing that \( V_L = L\frac{di}{dt} \), using KVL we get,

\[
V_C + iR + L\frac{di}{dt} = 0 \quad (24)
\]

From the above equation we get,

\[
\frac{di}{dt} = -\frac{V_C}{L} - \frac{iR}{L} \quad (25)
\]

And from the voltage across the capacitor,

\[
\frac{C}{dt} = \frac{i}{C} \quad (26)
\]

\[
\frac{dV_C}{dt} = \frac{i}{C} \quad (27)
\]

\[
\begin{pmatrix}
\frac{di}{dt} \\
\frac{dV_C}{dt}
\end{pmatrix} = \begin{pmatrix}
-\frac{R}{L} & -\frac{1}{L} \\
\frac{1}{C} & 0
\end{pmatrix}
\begin{pmatrix}
i \\
V_C
\end{pmatrix}
\]

(c) Find the eigenvalues of the matrix \( A \) and determine \( V_C(t) \) for \( t > 0 \). How are the transients going to behave for \( \frac{R}{2L} > \frac{1}{\sqrt{LC}} \) and \( \frac{R}{2L} < \frac{1}{\sqrt{LC}} \).

**Solution:**

To find the eigenvalues, we find the solutions to

\[
\det\left(\begin{array}{cc}
-\frac{R}{L} - \lambda & -\frac{1}{L} \\
\frac{1}{C} & -\lambda
\end{array}\right) = \lambda \left(\frac{R}{L} + \lambda\right) + \frac{1}{LC} = 0 \quad (28)
\]

This is a quadratic equation, so using the quadratic formula, we get

\[
\lambda = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}
\]

Let \( \lambda_0 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \) and \( \lambda_1 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \).
For simplicity, let $\alpha = \frac{R^2}{2L}$, $\gamma = \sqrt{\left(\frac{R^2}{2L}\right)^2 - \frac{1}{LC}}$. We have

$$V_C = c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t}$$  \hspace{1cm} (29)$$

$$i = c_0 C \lambda_0 e^{\lambda_0 t} + c_1 C \lambda_1 e^{\lambda_1 t}$$  \hspace{1cm} (30)$$

At $t = 0$, we substitute the initial conditions to get

$$V_s = c_0 + c_1$$  \hspace{1cm} (31)$$

$$0 = c_0 C \lambda_0 + c_1 C \lambda_1$$  \hspace{1cm} (32)$$

$$c_0 (1 - \frac{\lambda_0}{\lambda_1}) = V_s$$  \hspace{1cm} (33)$$

$$c_0 = -V_s \frac{\lambda_1}{2\gamma}, c_1 = V_s \frac{\lambda_0}{2\gamma}$$  \hspace{1cm} (34)$$

This gives us

$$V_C(t) = \frac{V_s}{2\gamma} e^{-\alpha t} \left(e^{-\gamma t} + e^{\gamma t}\right)$$

If $\frac{R^2}{2L} < \frac{1}{\sqrt{LC}}$, then $\gamma$ is complex and the transients are going to be sinusoidal in nature and the magnitude of oscillations will go down exponentially.

If $\frac{R^2}{2L} > \frac{1}{\sqrt{LC}}$, then $\gamma$ is real and the transients are going to exponentially die down.

(d) Determine the transfer function $\frac{V_L}{V_s}$ and find the roots of the denominator. Explain the similarity between the poles and eigenvalues of the above matrix.

**Solution:**

Using the KVL equation, we get

$$\frac{V_L}{V_s} = \frac{i(j\omega L)}{i(j\omega L + \frac{1}{j\omega C} + R)}$$

Roots of the denominator can be found by solving for the roots of the equation

$$(j\omega)^2 LC + j\omega RC + 1 = 0$$

$$-\omega^2 LC + j\omega RC + 1 = 0$$

Therefore, the roots of the quadratic equation are,

$$\omega = j \frac{R}{2L} \pm \sqrt{- \left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega = j \frac{R}{2L} \pm j \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$j\omega = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

The roots of the equation are the same as the eigenvalues. This is because the characteristic equation of the differential equation we formulated is the same as the denominator of the above transfer function. The poles of the transfer function thus determines the behavior of the transients of the above circuit. In addition, solving for the transfer function and getting the fourier inverse is an alternative process for solving for the transients.
7. RLC Circuit 2

Now consider the circuit shown below:

(a) Write down the KCL equation; express the total current through the source as a sum of currents in the capacitor and inductor.

Solution: Solving for the KCL at the node in the intersection between resistor and capacitor,

\[ i_s - i_C - i_L = 0 \]

\[ i_s = i_C + i_L \]

(b) Determine the equivalent impedance of the parallel circuit in the phasor domain.

Solution: From above,

\[ i_s - i_C - i_L = 0 \]

Current through capacitor, \( i_C = \frac{V_{out}}{j\omega C} \)

Current through inductor, \( i_L = \frac{V_{out}}{j\omega L} \)

Equivalent impedance,

\[ z_{par} = \frac{V_{out}}{i_s} \]
\[ = \frac{V_{out}}{i_s} + \frac{V_{out}}{j\omega L} \]
\[ = \frac{j\omega L}{1 - \omega^2 LC} \]

(c) Write down the transfer function of this circuit (take \( V_s \) as \( V_{in} \), and the voltage across the inductor as \( V_{out} \) as indicated in the circuit.)

Solution:

Knowing that

\[ V_s = V_{out} + V_R \]
\[ V_R = i_s R_s \]

The transfer function is given by,
\[
\frac{V_{out}}{V_s} = \frac{V_{out}}{V_{out} + i_s R_s} \tag{40}
\]

\[
\frac{V_{out}}{V_s} = \frac{V_{out}}{V_{out} + V_{out} \frac{R_s}{\text{par}}} \tag{41}
\]

\[
\frac{V_{out}}{V_s} = \frac{1}{1 + \frac{R_s(1-\omega^2 LC)}{j\omega L}} \tag{42}
\]

\[
= \frac{j\omega L}{j\omega L + R_s - R_s \omega^2 LC} \tag{43}
\]

(d) Sketch the bode plot for the above transfer function. What kind of filter is this circuit?

**Solution:**

As the transfer function goes to zero for both zero and infinite frequency, while it has high gain in a restricted band of frequencies, it’s a band pass filter.

Another important characteristic to consider about these circuits is the damping factor which is also given by the Q factor. Devices with high Q factor are underdamped and vice-versa. Q is related to the bandwidth and the central frequency.

\[
Q = \frac{\omega_0}{\Delta \omega} = R \sqrt{\frac{C}{L}}
\]

We can see the result in the exact magnitude of the transfer function, where with increased Q, the frequency response becomes narrower.

(e) Suppose the voltage source is removed from the circuit (make it an open circuit) at \( t \geq 0 \), i.e \( i_s = i_L = \frac{V_s}{R_s} \) for \( t < 0 \) and \( i_s = 0 \) for \( t \geq 0 \). Write down the differential equation in the form of the matrix

\[
\begin{pmatrix}
\frac{di_L}{dt} \\
\frac{dV_C}{dt}
\end{pmatrix} = A \begin{pmatrix} i_L \\ V_C \end{pmatrix} \tag{44}
\]

**Solution:** Applying KVL around the parallel circuit,

\[
V_L - V_C = 0
\]

\[
V_L = L \frac{di_L}{dt}
\]

\[
L \frac{di_L}{dt} + V_C = 0
\]
\[
\frac{di_L}{dt} = \frac{1}{L} V_C
\]

Also, the same current flows through the capacitor and the inductor. Therefore,

\[
C \frac{dV_C}{dt} = -i_L
\]

\[
\begin{pmatrix}
\frac{di_L}{dt} \\
\frac{dV_C}{dt}
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{L} \\
-\frac{1}{C} & 0
\end{pmatrix} \begin{pmatrix}
i_L \\
V_C
\end{pmatrix}
\]  

(45)

(f) Determine \(V_C(t)\) for \(t \geq 0\). What is the frequency of oscillation for the above circuit?

**Solution:** To find the eigenvalues, we find the solutions to

\[
\det \begin{pmatrix}
-\lambda & -\frac{1}{L} \\
\frac{1}{C} & -\lambda
\end{pmatrix} = \lambda^2 + \frac{1}{LC} = 0
\]

This is a quadratic equation, so using the quadratic formula, we get

\[
\lambda = \pm \sqrt{-\frac{1}{LC}}
\]

\[
\lambda = \pm j\omega_0, \quad \omega_0 = \sqrt{\frac{1}{LC}}
\]

\[
V_C = c_0 e^{j\omega_0 t} + c_1 e^{-j\omega_0 t}
\]

(46)

\[
i = c_0 C j\omega_0 e^{j\omega_0 t} - c_1 C j\omega_0 e^{-j\omega_0 t}
\]

(47)

At \(t = 0\), we substitute the initial conditions to get

\[
0 = c_0 + c_1
\]

(48)

\[
\frac{V_S}{R_S} = c_0 C j\omega_0 - c_1 C j\omega_0
\]

(49)

\[
c_0 = -c_1
\]

(50)

\[
2c_0 C j\omega_0 = \frac{V_S}{R_S}
\]

(51)

\[
c_0 = \frac{V_S}{2R_S C j\omega_0}
\]

(52)

This gives us

\[
V_C(t) = \frac{V_S}{2R_S C \omega_0} \sin(\omega_0 t)
\]

The oscillation frequency is thus equal to \(\omega_0\)
8. RLC circuit as passive filters

As originally conceived by Bode in the 1930s, Bode plot is only an asymptotic approximation of the frequency response, using straight line segments. It relies on using a logarithmic scale for the input frequency \(\omega\) to express the magnitude of the transfer functions in decibels (dB).

In this question, we will go through some examples to appreciate the beauty and simplicity of Bode plots. In the iPython notebook, you will see how well the approximation of Bode plots is in different regions. In particular, we will work with the RLC circuit shown below:

In the following questions, we will be exploring how to use the above RLC circuit to construct highpass, lowpass, and bandpass filters. As the name suggests, a highpass filter will suppress the low frequency components while keeping the high frequency components of the input unblocked. Since the circuit contains only passive elements, namely resistors, capacitors, and inductors, these filters are called passive filters. On the other hand, if the circuit contains op amps, transistors, or other active devices, it will become active filters.

(a) **Lowpass filter.** Treat \(V_s\) as the input and \(V_C\) as the output. Obtain the transfer function \(H_{LP} = \frac{V_C}{V_s}\), and its magnitude and phase. Draw the Bode plot for the magnitude. Explain why this is a lowpass filter.

**Solution:** To obtain the transfer function, we use the definition:

\[
H_{LP}(\omega) = \frac{V_C}{V_s} = \frac{(1/j\omega C)I}{V_s} = \frac{1}{(1 - \omega^2 LC) + j\omega RC}
\]

where we used \(V_s = I(R + j\omega L + \frac{1}{j\omega C})\), so \(\frac{V_s}{V_s} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}}\). For the magnitude:

\[
M_{LP}(\omega) = |H_{LP}(\omega)| = \frac{1}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2}}
\]

\[
\phi_{LP} = \begin{cases} 
-\pi - \tan^{-1}\left(\frac{\omega RC}{1 - \omega^2 LC}\right) & \omega > \frac{1}{\sqrt{LC}} \\
-\tan^{-1}\left(\frac{\omega RC}{1 - \omega^2 LC}\right) & \omega < \frac{1}{\sqrt{LC}} \\
-\frac{\pi}{2} & \omega = \frac{1}{\sqrt{LC}}
\end{cases}
\]

where we calculate the phase by first identifying which quadrant the complex transfer function lies in and then use arctan to get the angle.

For the bode plots, let \(\omega_0 = \frac{1}{\sqrt{LC}}\) and \(Q = \frac{\omega LC}{R}\), then we have:

\[
H_{LP}(\omega) = \frac{1}{(1 - (\omega/\omega_0)^2) + j\frac{\omega}{Q\omega_0}}
\]
where we recognize it is quadratic pole with $N = 1$, $\xi = \frac{1}{2Q}$. Using the functional form, we have:

It is a low pass filter because the transfer has large magnitude when the frequency is low (on the left side of the graph), and the magnitude is reducing exponentially as the frequency is above the resonance frequency $\omega_0$.

If we change the resistance $R$, you can see that it doesn’t change the resonance frequency $\omega_0$, but it does change $Q = \frac{\omega_0 L}{R}$, which is known as the quality factor. As we increase it, we have more dramatic peaks around the resonance frequency, and it becomes more pronounced relative to the level of the asymptotes. In the plot, we show several cases for the $Q$, you can see that for large $Q$, the asymptotic approximation by bode plots are less accurate especially around the resonance frequency.

(b) **Highpass filter**. Let $V_L$ be the output. Obtain the transfer function $H_{HP} = \frac{V_L}{V_s}$, and its magnitude and phase. Draw the Bode plot for the magnitude. Explain why this is a highpass filter.

**Solution**: To obtain the transfer function, we use the definition:

$$H_{HP}(\omega) = \frac{V_L}{V_s} = \frac{j\omega LI}{V_s} = \frac{-\omega^2 LC}{(1 - \omega^2 LC) + j\omega RC}$$

where we used $V_s = I(R + j\omega L + \frac{1}{j\omega C})$, so $\frac{I}{V_s} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}}$. For the magnitude:

$$M_{HP}(\omega) = |H_{HP}(\omega)| = \frac{\omega^2 LC}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2}}$$

$$\phi_{HP} = \begin{cases} 
-\tan^{-1}\left(\frac{\omega RC}{1 - \omega^2 LC}\right) & \omega > \frac{1}{\sqrt{LC}} \\
\pi - \tan^{-1}\left(\frac{\omega RC}{1 - \omega^2 LC}\right) & \omega < \frac{1}{\sqrt{LC}} \\
\frac{\pi}{2} & \omega = \frac{1}{\sqrt{LC}}
\end{cases}$$

where we calculate the phase by first identifying which quadrant the complex transfer function lies in and then use arctan to get the angle.

$$\phi_{HP} = \pi - \tan^{-1}\left(\frac{\omega RC}{1 - \omega^2 LC}\right)$$

For the bode plots, let $\omega_0 = \frac{1}{\sqrt{LC}}$ and $Q = \frac{\omega_0 L}{R}$, then we have:

$$H_{LP}(\omega) = \frac{-(\omega/\omega_0)^2}{(1 - (\omega/\omega_0)^2) + j\omega/\omega_0}$$
where we recognize it is composed of a quadratic pole with \( N = 1, \xi = \frac{1}{2Q} \) and a Zero @ Origin with \( N = 2 \) and a constant with \( K = \frac{1}{\omega_0^2} \). Using the functional form, we have:

It is a high pass filter because the transfer has large magnitude when the frequency is high (on the right side of the graph), and the magnitude is reducing exponentially as the frequency is less than the resonance frequency \( \omega_0 \).

Similar to the previous part, if we increase \( Q \), we have more dramatic peaks around the resonance frequency, and it becomes more pronounced relative to the level of the asymptotes.

(c) **Bandpass filter.** How can you obtain a bandpass filter based on your findings above? Write out the transfer function and its magnitude and phase.

**Solution:** Yes we can. The output will be \( V_R \) to construct a bandpass filter:

\[
H_{BP}(\omega) = \frac{V_R}{V_s} = \frac{RI}{(1 - \omega^2LC) + j\omega RC}
\]

where we used \( V_s = I(R + j\omega L + \frac{1}{j\omega C}) \), so \( \frac{V_s}{I} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}} \).

\[
M_{BP}(\omega) = |H_{BP}(\omega)| = \frac{\omega RC}{\sqrt{(1 - \omega^2LC)^2 + \omega^2R^2C^2}}
\]

\[
\phi_{LP} = \begin{cases} 
\frac{-\pi}{2} - \tan^{-1} \left( \frac{\omega RC}{1 - \omega^2LC} \right) & \omega > \frac{1}{\sqrt{LC}} \\
\frac{\pi}{2} - \tan^{-1} \left( \frac{\omega RC}{1 - \omega^2LC} \right) & \omega < \frac{1}{\sqrt{LC}} \\
0 & \omega = \frac{1}{\sqrt{LC}} 
\end{cases}
\]

where we calculate the phase by first identifying which quadrant the complex transfer function lies in and then use arctan to get the angle.

(d) The **resonant frequency**, \( \omega_0 \), is the input frequency (other than 0 and \( \infty \)) that leads to the elimination of the imaginary part of the circuit impedance, i.e., the impedance is purely real. Find the resonant frequency for the RLC circuit above.

**Solution:**

The impedance of the circuit as measured from both sides of the voltage source is given by:

\[
Z_R + Z_L + Z_C = R + j\omega L + \frac{1}{\frac{1}{j\omega C}} = R + j(\omega L - \frac{1}{\omega C})
\]

which is the same for all the lowpass, highpass, and bandpass filters. To eliminate the imaginary part, we can set:

\[
\omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \omega_0 = \sqrt{\frac{1}{LC}} \tag{54}
\]

which, by definition, is the resonant frequency.
For mobile communications, the center frequency is approximately 800 MHz. In the IPython notebook, experiment with different \(L\) and \(C\) to center the bandpass filter.

**Solution:**
To center the bandpass filter, we need to find \(L\) and \(C\) such that \(\omega_0 = \sqrt{\frac{1}{LC}} = 8 \times 10^8\). Please refer to the IPython notebook for suitable values.

You might have noticed that the advantage of Bode plot is that it makes it easier to work with transfer functions that have multiple factors, known as functional forms. We can cast \(H(\omega)\) into the **standard form**:

\[
H(\omega) = A_1(\omega)A_2(\omega)\ldots A_n(\omega)
\]

where \(A_1\) to \(A_n\) assume one of the possible functional forms below:

- **Constant factor:** \(H = K\)
- **Zero @ origin:** \(H = (j\omega)^N\)
- **Pole @ origin:** \(H = \frac{1}{(j\omega)^N}\)
  - **Simple zero:** \(H = (1 + j\omega/\omega_c)^N\)
  - **Simple pole:** \(H = \frac{1}{(1 + j\omega/\omega_c)^N}\)
  - **Quadratic zero:** \(H = (1 + j2\zeta\omega/\omega_c + (j\omega/\omega_c)^2)^N\)
  - **Quadratic pole:** \(H = (1 + j2\zeta\omega/\omega_c + (j\omega/\omega_c)^2)^{-N}\)

The construction thus becomes simple addition or subtraction of the functional forms. For instance, \(H(\omega) = 10^{\frac{1}{1+(j\omega/\omega_p)}}\), where \(A_1 = 10\), \(A_2 = 1 + j\omega/\omega_Z\), \(A_3 = \frac{1}{1+j\omega/\omega_p}\).

**Solution:**
For transfer function \(H(\omega) = M(\omega)e^{j\phi(\omega)}\), how to represent the magnitude \(M(\omega)\) and phase \(\phi(\omega)\) with the magnitudes \(|A_1(\omega)|\) and phase \(\phi_{A_1}(\omega)\)?

Since we have \(H(\omega) = A_1(\omega)A_2(\omega)\ldots A_n(\omega)\), and \(A_i(\omega) = |A_i(\omega)|e^{j\phi_{A_i}(\omega)}\) in the polar representation, we have:

\[
H(\omega) = M(\omega)e^{j\phi(\omega)} = |A_1(\omega)||A_2(\omega)|\cdots |A_n(\omega)|e^{j(\phi_{A_1}(\omega)+\phi_{A_2}(\omega)+\cdots+\phi_{A_n}(\omega))}
\]

Therefore, we have:

\[
M(\omega) = |A_1(\omega)||A_2(\omega)|\cdots |A_n(\omega)|
\]
\[
\phi(\omega) = \phi_{A_1}(\omega) + \phi_{A_2}(\omega) + \cdots + \phi_{A_n}(\omega)
\]

by comparison.

**Solution:**
Consider the transfer function

\[
H(\omega) = \frac{(j10\omega + 30)^2}{(300 - 3\omega^2 + j90\omega)}
\]

Refer to the IPython notebook for constructing the Bode plots using functional forms.

**Solution:**
In addition to the Bode plots, we also plotted the magnitude of the transfer functions without approximations. Please comment on the differences.
We can see that Bode plots are linear approximations of the magnitude of the transfer functions. The approximation is generally good at regions without corners. Around the corners, the exact plot usually have some “shootings” or smooth curves.

This homework problem addresses the essential concepts of Bode plots. Please check the following table to improve your understanding.

### 9. Your Own Problem

Write your own problem related to this week’s material and solve it. You may still work in groups to brainstorm problems, but each student must submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

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