Background
As discussed previously, we know the following about observability:
\[
\dot{\mathbf{x}}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t) \\
\mathbf{y}(t) = C\mathbf{x}(t)
\]
in which we are given \(\mathbf{x}_0\) and \(C, CA, CA^2, \ldots, CA^{n-1}\).

If rank \(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}\) = \(n\), then the system is observable.

We take this example discussed from a previous lecture, and think about it in terms of observation:
\[
\dot{\mathbf{x}}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t) \\
\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

Let \(C\) be the observation matrix of the first state \(\begin{bmatrix} 1 & 0 \end{bmatrix}\), then \(CA = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}\)...

\(C\) and \(CA\) are linearly independent.
Let us try another check of this matrix as a more explicit example:

\[
\begin{bmatrix}
  a \\
  b
\end{bmatrix}
\]

\[
y(0) = a
\]

\[
\begin{bmatrix}
  1 & 1 \\
  0 & 2
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{bmatrix}
a + b \\
2b
\end{bmatrix}
\]

\[
y(1) = a + b
\]

Question: If you know the initial condition and evolution matrix and inputs of a system, do you need to observe anything at all? Mathematically, there is no need to observe anything. However, realistically, you definitely need observations, and use those to develop a controls system.

Main Topic of Today

Let’s say we have a system:

\[
\begin{aligned}
\bar{x}(t + 1) &= A\bar{x}(t) + Bu(t) \\
\end{aligned}
\]

in which A is a matrix and B is a vector.

We know that this system is stable in open loop if all the eigenvalues of A are inside the unit circle, |\lambda| < 1.

Open Loop vs. Closed Loop: Block Diagram Views

![Open Loop System Diagram](image)

![Closed Loop System Diagram](image)

In an open loop system, inputs are chosen without looking at the state of the system. In a closed loop system, the input is chosen with the feedback of the current state of the system.
With this given, we can take a system that is unstable with eigenvalues wherever and you can use matrix K to move all $\lambda$ to wherever you want.

In the context of an example: $\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

Upon looking at the matrix A, we see it is a triangular matrix and thus the eigenvalues lie on the diagonal. Therefore, the eigenvalues of A are 1 and 2, and thus it does not satisfy our earlier criteria of stability.

We choose K so the solution satisfies stability. Let $u(t) = K \vec{x}(t)$ and $K = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$

$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}(t)$

$\vec{x}(t+1) = \left( \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \right) \vec{x}(t)$

$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ f_1 & 2 + f_2 \end{bmatrix} \vec{x}(t)$

From this, we can find an appropriate choice of $f_1$ and $f_2$, allowing you to move eigenvalues to wherever you want.

$$\det \left( \begin{bmatrix} 1 & 1 \\ f_1 & 2 + f_2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = (1-\lambda)(2+f_2-\lambda) - f_1 = \lambda^2 + (-3-f_2)\lambda + (2+f_2-f_1)$$

We can set $\lambda^2 + (-3-f_2)\lambda + (2+f_2-f_1)$ to whatever polynomial you want, then solve the system of equations for $f_1, f_2$.

So, if you say, “I want $\lambda_1, \lambda_2$ as the eigenvalues”

Then we set $\lambda^2 + (-3-f_2)\lambda + (2+f_2-f_1)$ equal to the polynomial generated by the desired eigenvalues.

$$\lambda^2 + (-3-f_2)\lambda + (2+f_2-f_1) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + (-\lambda_1 - \lambda_2)\lambda + \lambda_1 \lambda_2$$

Solve:

$$-3-f_2 = -\lambda_1 - \lambda_2$$
$$2+f_2-f_1 = -\lambda_1 \lambda_2$$
For $\lambda_1 = 1/2, \lambda_2 = 0$, we find that $f_1 = -5/2, f_2 = -1/2$

Check:

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1/2 & -5/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/2 & -1/2 \end{bmatrix}$$

is stable