Consider system:
\[
\mathbf{x}(t + 1) = a\mathbf{x}(t) + b\mathbf{u}(t) + \mathbf{\omega}(t)
\]
where \(\mathbf{\omega}(t)\) is a disturbance in the system and \(\mathbf{x}(0) = \mathbf{x}_o\). To what extent does this disturbance cause trouble?

**Open loop case:**
\(\mathbf{u}(t) = 0\), System is stable iff \(|a| < 1\)

Solution:
\[
\mathbf{x}(t) = a^t \mathbf{x}_o + \sum_{\tau=0}^{t-1} a^{t-1-\tau} \mathbf{\omega}(\tau)
\]
Without loss of generality, let \(\mathbf{x}_o = 0\) and \(i = t - 1 - \tau\)
\[
\mathbf{x}(t) = a^t \mathbf{x}_o + \sum_{i=0}^{t-1} a^i \mathbf{\omega}(t - 1 - i)
\]
Assume \(|\mathbf{\omega}(t)| \leq \epsilon\) (disturbance is bounded by some value \(\epsilon\)):
\[
|\mathbf{x}(t)| \leq \sum_{i=0}^{t-1} |a|^i |\mathbf{\omega}(t - 1 - i)| \leq \sum_{i=0}^{t-1} |a|^i \epsilon \leq \sum_{i=0}^{\infty} |a|^i \epsilon
\]
Using the sum of a geometric series (if \(a < 1\)):
\[
\sum_{i=0}^{\infty} |a|^i \epsilon = \frac{\epsilon}{1 - |a|}
\]
What if \(a > 1\)? \(\Rightarrow\) System is unstable.
Does there exist a sequence $\vec{\omega}(t)$ (where $|\vec{\omega}(t)| \leq \epsilon$) such that $|\vec{x}(t)| \to \infty$?

In general, $\vec{\omega}(t) = \frac{e^t}{|a|}$ works.

In the vector case

$$\vec{x}(t + 1) = A\vec{x}(t) + \vec{\omega}(t)$$

A bounded distribution has $|\vec{\omega}(t)[j]| \leq \epsilon$ for all $t, j$.

If $A$ is diagonalizable for all $V$ such that $A = V\Lambda V^{-1}$ where $\Lambda$ is a diagonal matrix and each eigenvalue $\lambda$ in this matrix exists such that $|\lambda| < 1$:

Change coordinates to

$$\vec{z}(t) = V^{-1}\vec{x}(t)$$

So,

$$\vec{z}(t + 1) = \Lambda \vec{z}(t) + V^{-1}\vec{\omega}(t)$$

The first term, $\Lambda \vec{z}(t)$, is bounded by the $\lambda$ values in the diagonal matrix, and the term $V^{-1}\vec{\omega}(t)$ is bounded by $\epsilon$ and the maximum entries of $V^{-1}$.

$\to$ Bounded input implied bounded output if the matrix $A$ is diagonalizable.

What if the matrix $A$ is not diagonalizable?

There exists a matrix $Q$ of orthonormal vectors such that

$$A = Q \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & stuff & \\ & & \lambda_3 & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} Q^{-1}$$

This is known as Schur Decomposition.

If we change coordinates to $\vec{z}(t) = Q^{-1}\vec{x}(t)$,
\[ \vec{z}(t + 1) = \begin{bmatrix} \lambda_1 & stuff \\ \lambda_2 & \lambda_3 \\ 0 & \ddots \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ \vdots \\ \vec{z}_n(t) \end{bmatrix} + Q^{-1}\vec{\omega} \]

\[ \vec{z}_n(t + 1) = \lambda \vec{z}_n(t) + \vec{\omega}(t) \]

We know \( |\vec{z}_n(t)| < C_n \epsilon \) where \( C_n \) is a constant obtained from the “stuff” part of the matrix.

\[ \vec{z}_{n-1}(t + 1) = \lambda_{n-1}\vec{z}_{n-1}(t) + "junk"\vec{z}_n(t) + \omega_{n-1}(t) \]

All three terms of the output are bounded, again showing that a bounded input produces a bounded output.