EE16B Designing Information Devices and Systems II **Spring 2016**

Lecture Notes: 26

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26.1 Non-linear discrete system

Consider the following non-linear discrete system

$$x(t+1) = x(t)^2 + u(t)$$

The system will keep squaring itself. We know that when the input is 0 the system is stable. If our input is within [-1,1] the system will converge to 0 and will be stable. If the input is outside the range, the system will blow up.

Assume we want to control the system and hold it at x_{target}. We want to find the input such that

$$x_{target} = x_{target}^2 + u_{target}$$

$$u_{target} = x_{target} - x_{target}^2$$

For an example lets choose $x_{target} = 1$, $u_{target} = 0$. Let $z(t)$ be defined as how far we are from the target

$$z(t) = x(t) - 1$$

We want a linear model for z.

$$\begin{aligned} z(t+1) = x(t+1) - 1 &= x(t)^2 + u(t) - 1 \\ &= (z(t)+1)^2 + u(t) - 1 \\ &= 2z(t) + u(t) + z(t)^2 \end{aligned}$$

The $z(t)^2$ is the disturbance. We expect $z(t)$ to be small.

We want to have x in the neighborhood of 1 and we restrict our attetntion such that $|z(t)| < \delta$.

$$z(t+1) = 2z(t)$$

This equation itself is not stable because 2 is outside the unit circle and the system blows up. Hence, we want to apply a control that stabilizes the system fast.

We pick an arbitrary controller

$$u(t) = -1.75(x(t) - 1) = -1.75z(t)$$

We then have a closed-loop form

$$z(t+1) = \frac{1}{4}z(t) + w(t)$$

The gain on the disturbance can be found using geometric sum

$$\frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

We want to check if the disturbance is bounded. If $|z(t)| < \delta$ then $|w(t)| < \delta^2$

For example we choose $\delta = \frac{1}{2}$ then we have $\delta^2 = \frac{1}{4}$. We see that the disturbance can move the system by

$$\frac{4}{3}\delta^2 = \frac{1}{3}$$

Which is indeed smaller than δ, $\frac{1}{2}$.

The δ in real life denotes the operation circle that we would like to work in. We will say that the system is broken if the system is outside of δ. Hence, we want to choose a system such that we are guaranteed to stay within the stable range.

Now, let's assume a new system with external disturbance

$$x(t+1) = x(t)^2 + u(t) + x_{ext}(t)$$

Where $x_{ext}(t)$ is the external disturbance. We want to figure out how much external disturbance we can tolerate with the current system.

We assume $|x_{ext}(t)| < \epsilon$

$$\frac{4}{3}(\frac{1}{4} + \epsilon) < \frac{1}{2}$$
$$\epsilon = \frac{1}{8}$$

With the current control law, with $\frac{1}{8}$ external disturbance the system is still stable.

26.2 System bound revisited

Theorem If all $\lambda(A)$, $|\lambda| < 1$, then if $\vec{w}$ is bounded so $|\vec{w}(t)[j]| < \epsilon$, then $\exists K > 0$ s.t $|\vec{x(t)}[j]| < K\epsilon$.

The proof was discussed previously and it relies on the fact that the matrix is upper triangulable. Then the last entry in the upper triangle matrix is bounded by some constant. The bounded output is then fed in the next equation above as the input and we also get a bounded output. This relationship recurses all the way through the system and we see that there exists some constant K.