38.1 Polynomial Interpolation and DFT

Choose evaluation points $s_0, s_1...s_n$ with $|s_j| = 1$ s.t evaluating $s_j^n$ won’t blow up. Having a magnitude 1 will neither shrink nor blow up the polynomial.

Pick $n^{th}$ root of unity. $s^n_h = 1$.

$\begin{align*}
s_0 &= e^{i \frac{2\pi}{n}} \\
\dot{s}_h &= e^{i \frac{2\pi}{n}h} \\
s_{n-1} &= e^{i \frac{2\pi}{n}(n-1)}
\end{align*}$

Evaluate polynomial at these points. Polynomials are linear combinations of vectors $t^0, t^1...t^{n-1}$ which we can treat as basis vectors $\vec{b}_0, \vec{b}_1...\vec{b}^{n-1}$.

$\vec{b}_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 1 \\ e^{i \frac{2\pi}{n}} \\ e^{i \frac{2\pi}{n}2} \\ \vdots \\ e^{i \frac{2\pi}{n}(n-1)} \end{bmatrix}, \vec{b}_h = \begin{bmatrix} 1 \\ e^{i \frac{2\pi h}{n}} \\ e^{i \frac{2\pi h2}{n}} \\ \vdots \\ e^{i \frac{2\pi h(n-1)}{n}} \end{bmatrix}$

Note that these vectors can be seen as the DFT basis.

If we know that $\vec{x}$ has the form

$\vec{x} = \sum_{j=0}^{k-1} \alpha_j \vec{u}_j$

Then, $\vec{x}$ can be recovered from any $k$ samples of $\vec{x}$.

One problem with this form is that it will always be complex, but our signals are real. So we need a more general result.

Consider

$x(t) = \alpha_0 t^a + \alpha_1 t^{a+1} + ... + \alpha_k t^{a+k-1}$

Where $a$ is known. We can show $x(t)$ is like a polynomial.

$x(t) = t^a(\alpha_0 + \alpha_1 t^1 + ... + \alpha_k t^{k-1})$
\[ x(t) = t^a \tilde{x}(t) = t^a \sum_{j=0}^{k-1} \alpha_j t^j \]

Given \( k \) samples of \((t_i, y_i = x(t_i))\), can we reconstruct a point at \( x(t) \)? Moreover, given the sample of \( x \) can we get samples of \( \tilde{x} \)? If we know what \( \tilde{x} \) is, we can reconstruct \( x \). So what we want is \( \tilde{y}_i = \tilde{x}(t_i) \).

\[
\begin{align*}
y_i &= x(t_i) = t^a_i \tilde{x}(t_i) = t^a_i \tilde{y}_i \\
\tilde{y}_i &= \frac{y_i}{t^a_i} \\
\tilde{x}(t) &= \sum_{i=0}^{k=1} \tilde{y}_i l_i(t)
\end{align*}
\]

This means

\[
x(t) = t^a \tilde{x}(t) = \sum_{i=0}^{k-1} t^a_i \tilde{y}_i l_i(t) = \sum_{i=0}^{k-1} y_i t^a_i l_i(t)
\]

Hence we can see that

If \( \tilde{x} = \sum_{j=0}^{k-1} \alpha_j \tilde{u}_{j+a} \), then \( \tilde{x} \) can be recovered from any \( k \) samples.

In particular,

If \( \tilde{x} = \sum_{j=-m}^{m} \alpha_j \tilde{u}_j \), then \( \tilde{x} \) can be recovered from \( 2m + 1 \) samples.