Notes

Polynomial Interpolation

Given $n+1$ distinct points, we can find a unique degree n polynomial that passes through these points. Let the polynomial p be,

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$$

Let the $n+1$ points be,

$$p(x_0) = y_0, p(x_1) = y_1, \cdots, p(x_n) = y_n$$

Where,

$$x_0 \neq x_1 \neq \cdots \neq x_n$$

We can construct a matrix as follows to recover the polynomial p.

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Zero-Order Hold

The zero-order hold is another method of interpolation where instead of using a polynomial to fit the data, we just extrapolate the value of $y(t)$, where t is a sample time, and hold it until $t+T$ where T is the time between sample points. Then $y(t+T)$ is the next sample of y and is held until $t+2T$. This is continued for all sample points. This eventually creates a piecewise function that looks somewhat like a staircase.

Questions

1. **Lagrange interpolation and polynomial basis**

   In practice, to approximate some unknown or complex function $f(x)$, we take n evaluations/samples of the function, denoted by $\{(x_i, y_i \triangleq f(x_i));\ 0 \leq i \leq n-1\}$. With the Occam's razor principle in mind, we try to fit a polynomial function of least degree (which is $n-1$) that passes through all the given points.

   (a) Using the polynomial basis $\{1, x, x^2, \cdots, x^{n-1}\}$, the fitting problem can be cast into finding the coefficients $a_0, a_1, \cdots, a_{n-1}$ of the function

   $$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

such that $g(x_i) = y_i$, $\forall i = 0, 1, \cdots n-1$. Find out the set of equations that need to be satisfied, and write them in a matrix form $A\vec{a} = \vec{y}$, with $\vec{a} = [a_0, a_1, \cdots, a_{n-1}]^T$ and $\vec{y} = [y_0, y_1, \cdots, y_{n-1}]^T$

(b) Now we observe that in order to find those coefficients, we need to calculate $\vec{a} = A^{-1}\vec{y}$. The matrix inversion is computationally expensive and numerically inaccurate when n is large. The idea of Lagrange interpolation is to use a different set of basis $\{L_0(x), L_1(x), \cdots, L_{n-1}(x)\}$, which has the property that

$$L_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

With that the fitting problem becomes finding the coefficients $b_0, b_1, \cdots, b_{n-1}$ of the function

$$h(x) = b_0 L_0(x) + b_1 L_1(x) + b_2 L_2(x) + \cdots + b_{n-1} L_{n-1}(x)$$

such that $h(x_i) = y_i$, $\forall i = 0, 1, \cdots n-1$. Again, find out the set of equations that need to be satisfied, and write them in a matrix form. What do you observe?

(c) Show that if we define

$$L_i(x) = \Pi_{j=0; j\neq i}^{j=n-1} \frac{(x - x_j)}{(x_i - x_j)}$$

then the property required in part (b) is satisfied. What is the intuition behind this construction?

(d) Based on the previous two parts, write down the explicit form of $h(x)$ with the samples $\{(x_i, y_i); 0 \leq i \leq n-1\}$. The resulting formula is the so called Lagrange polynomial which passes through the n sampled points.

(e) Find the Lagrange polynomial given evaluated samples $f(-1) = 3, f(0) = -4, f(1) = 5, f(2) = -6$.

2. Zero-Order Hold

Consider the following voltage samples:

Time (t)	Voltage (V)
0	0
0.25	0.48
0.5	0.84
0.75	1
1	0.9
1.25	0.6
1.5	0.14
1.75	-0.35
2	-0.75
2.25	-0.97
2.5	-0.95
2.75	-0.7

(a) Draw the zero-order hold interpolation of the samples

(b) What type of function is this?

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