# Notes

**The Natural Continuous Inner Product**

We can define the following inner product on square integrable functions. Let $f$ and $g$ be such functions. Then,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

# Questions

1. **Projecting functions**

   Let $\{e_k\}_{k\in\mathbb{Z}}$ be a set of orthonormal functions and let $f$ be an arbitrary square integrable function. Let $\mathscr{E}$ be the linear subspace spanned by $\{e_k\}_{k\in\mathbb{Z}}$.

   (a) Prove that $\{e_k\}_{k\in\mathbb{Z}}$ are linearly independent.

   (b) What is the projection of $f$ onto the subspace $\mathscr{E}$?

2. **Haar Basis**

   Consider the following function,

   $$\phi(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{otherwise} \end{cases}$$

   Define,

   $$\phi_k^n = 2^{\frac{n}{2}}\phi(2^n x - k)$$

   (a) Intuitively speaking, the support of a function $f$ is defined as the set of points $x$ such that $f(x) \neq 0$. What is support of the function $\phi_k^0$?

   (b) Prove that $\{\phi_k^0\}_{k\in\mathbb{Z}}$ forms an orthonormal set.

   (c) What is the support of $\phi_k^n$?

   (d) Prove that $\{\phi_k^n\}_{k\in\mathbb{Z}}$ forms an orthonormal set.

Observe that $\{\phi_k^n\}_{k\in\mathbb{Z}}$ forms an orthonormal set with $n$ determining the resolution, or fineness, of any arbitrary square integrable function that the orthonormal set can represent since the support gets finer as $n$ increases. Further more, if we define,

$$V_n = \operatorname{span}\left(\{\phi_k^n\}_{k\in\mathbb{Z}}\right),$$

we have the property that,

$$V_n \subset V_{n+1}$$

Let us explore this.

(e) Find coefficients $p_l$ such that,

$$\phi_k^0 = \sum_{l\in\mathbb{Z}} p_l \phi_l^1$$

(f) Using the previous answer, find a function $\psi_k^0$ with the **same support** as $\phi_k^0$ such that,

$$\langle \phi_k^0, \psi_k^0 \rangle = 0$$

*Hint.* Use the coefficients from the previous function.

**Contributors:**

- Siddharth Iyer.