EECS 16B Designing Information Devices and Systems II
Spring 2017 Murat Arcak and Michel Maharbiz **Discussion 13A**

1 Notes

1.0.1 Discrete Fourier Transform

Assume we are working with an N length discrete signal and we would like to find its discrete frequencies. This is done through the Discrete Fourier Transform (DFT), which is simply a change of basis to what is called the DFT basis.

First, let us vectorize our signal. If $x[n]$ is our input signal, we model it as a vector by letting the n^{th} coordinate by $x[n]$. In other words,

$$\vec{x} = [x[0], x[1], x[2], \ldots, x[N-1]]^T$$

In order to decompose $\vec{x}$ into its constituent frequencies, we must find the vector representation of these frequencies.

Given that we have an N length signal, we have N different discrete frequencies of the following form.

$$u_k[t] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kt} \text{ for } k = 0, 1, \ldots N-1$$

To simplify we let

$$W_k = e^{jk\frac{2\pi}{N}}$$

and we rewrite

$$u_k[t] = \frac{1}{\sqrt{N}} W_k^t \text{ for } k = 0, 1, \ldots N-1$$

In building up a frequency basis, we vectorize the above frequencies in a manner similar to how we vectorized $\vec{x}$. Define $\vec{u}_k$ as follows.

$$\vec{u}_k = \frac{1}{\sqrt{N}} \left[1, W_k, W_k^2, \ldots, W_k^{N-1}\right]^T$$

$\{\vec{u}_k\}_{k=0}^{N-1}$ is an orthonormal set of vectors. To see why, first consider arbitrary $\vec{u}_p$ and $\vec{u}_q$ such that $p \neq q$.

$$\begin{aligned}
\langle \vec{u}_p, \vec{u}_q \rangle &= \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}pn} e^{j\frac{2\pi}{N}qn} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(q-p)n}
\end{aligned}$$

Before we continue, let us remind ourselves about the sum of a finite geometric series. Let S be the sum of the series. Then,

$$S = 1 + a + a^2 + \cdots a^{N-1}$$

Then,

$$aS = a + a^2 + a^3 + \cdots a^N$$

Subtracting the two, we get,

$$(1-a)S = 1 - a^N \implies S = \frac{1-a^N}{1-a}$$

Applying this, we get,

$$\begin{aligned}
\langle \vec{u}_p, \vec{u}_q \rangle &= \frac{1}{N} \sum_{n=0}^{N-1} \Big(\underbrace{e^{j\frac{2\pi}{N}(q-p)}}_{a} \Big)^n \\
&= \frac{1}{N} \left(\frac{1-a^N}{1-a} \right) \\
&= \frac{1}{N} \left(\frac{1-e^{j\frac{2\pi}{N}(q-p)N}}{1-e^{j\frac{2\pi}{N}(q-p)}} \right)
\end{aligned}$$

Note that $q-p$ is an non-zero integer. This means that,

$$e^{j\frac{2\pi}{N}(q-p)N} = e^{j2\pi(q-p)} = 1$$

Applying this, we get,

$$\langle \vec{u}_p, \vec{u}_q \rangle = 0$$

Finally, we also observe that, for a particular DFT basis vector,

$$\langle \vec{u}_p, \vec{u}_p \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}pn} e^{j\frac{2\pi}{N}pn} = \frac{1}{N} \sum_{n=0}^{N-1} 1 = 1$$

Thus, $\{\vec{u}_k\}_{k=0}^{N-1}$ is an orthonormal set of vectors and is a valid basis. The coefficients of $\vec{x}$ within this basis are called the frequency components of $\vec{x}$ and are often denote by $\vec{X}$.

$$\vec{X} = [\langle \vec{x}, \vec{u}_0 \rangle, \langle \vec{x}, \vec{u}_1 \rangle, \cdots, \langle \vec{x}, \vec{u}_{N-1} \rangle]^T$$

The k^{th} frequency component is the k^{th} coordinate of $\vec{X}$ and is denoted as $X[k]$. If we want to get the component in the same space as $\vec{x}$, we complex the projection.

$$\text{proj}_{\vec{u}_k} \vec{x} = X[k] \vec{u}_k$$

It is worthwhile to note that there is a conjugate property we often exploit.

$$e^{j\frac{2\pi}{N}pn} = e^{-j\frac{2\pi}{N}(N-p)n}$$

This means that,

$$\vec{u}_k = \overline{\vec{u}_{N-k}}$$

Finally, since we are using complex exponentials, there is a periodicity that can be succinctly expressed with the remainder operation (also called mod). Let p be any arbitrary integer.

$$\vec{u}_p = \vec{u}_{p \bmod N}$$

2 Questions

1. Roots of Unity

The DFT is a coordinate transformation to a basis made up of roots of unity. In this problem we explore some properties of the roots of unity. An Nth root of unity is a complex number z satisfying the equation $z^N = 1$ (or equivalently $z^N - 1 = 0$).

(a) Show that $z^N - 1$ factors as

$$z^N - 1 = (z-1)(\sum_{k=0}^{N-1} z^k).$$

(b) Show that any complex number of the form $\omega_k = e^{j\frac{2\pi}{N}k}$ for $k \in \mathbb{Z}$ is an N-th root of unity.

(c) Draw the fifth roots of unity in the complex plane. How many of them are there?

(d) Let $\omega_1 = e^{j\frac{2\pi}{5}}$. What is ω_1^2? What is ω_1^3? What is ω_1^{42}?

(e) What is the complex conjugate of ω_1? What is the complex conjugate of ω_{42}?

(f) Compute $\sum_{k=0}^{N-1} \omega^k$ where ω is some root of unity. Does the answer make sense in terms of the plot you drew?

2. DFT of pure sinusoids We can think of a real-world signal that is a function of time $x(t)$. By recording its values at regular intervals, we can represent it as a vector of discrete samples $\vec{x}$, of length n.

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \tag{1}$$

Let $\vec{X} = \begin{bmatrix} X[0] & \dots & X[N-1] \end{bmatrix}^T$ be the signal $\vec{x}$ represented in the frequency domain, that is

$$\vec{X} = F^{-1}\vec{x} = F^*\vec{x} \tag{2}$$

where F is a matrix of the DFT basis vectors ($\omega = e^{j\frac{2\pi}{N}}$).

$$F = \begin{bmatrix} | & & | \\ \vec{u}_0 & \cdots & \vec{u}_{n-1} \\ | & & | \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_1 & W_2 & \cdots & W_{N-1} \\ 1 & W_1^2 & W_2^2 & \cdots & W_{N-1}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_1^{N-1} & W_2^{N-1} & \cdots & W_{N-1}^{N-1} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix} \tag{3}$$

Alternatively, we have that $\vec{x} = F\vec{X}$ or more explicitly

$$\vec{x} = X[0]\vec{u}_0 + \cdots + X[n-1]\vec{u}_{n-1} \tag{4}$$

In other words, $\vec{x}$ is a linear combination of the complex exponentials $\vec{u}_i$ with coefficients $X[i]$.

(a) Consider the continuous-time signal $x(t) = \cos\left(\frac{2\pi}{3}t\right)$. Suppose that we sampled it every 1 second to get (for $n = 3$ time steps):

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) \end{bmatrix}^T.$$

Compute $\vec{X}$ and the basis vectors $\vec{u}_k$ for this signal.

(b) Now for the same signal as before, suppose that we took $n = 6$ samples. In this case we would have:

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) & \cos\left(\frac{2\pi}{3}(3)\right) & \cos\left(\frac{2\pi}{3}(4)\right) & \cos\left(\frac{2\pi}{3}(5)\right) \end{bmatrix}^T.$$

Repeat what you did above. What are $\vec{X}$ and the basis vectors $\vec{u}_k$ for this signal.

(c) Let's do this more generally. For the signal $x(t) = \cos\left(\frac{2\pi k}{N}t\right)$, compute $\vec{X}$ of its vector form in discrete time, $\vec{x}$, of length $n = N$:

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi k}{N}(0)\right) & \cos\left(\frac{2\pi k}{N}(1)\right) & \cdots & \cos\left(\frac{2\pi k}{N}(n-1)\right) \end{bmatrix}^T.$$

Contributors:

- Siddharth Iyer.
- John Maidens.
- Yen-Sheng Ho.
- Yu-Yun Dai.