Notes

Linear Time-Invariant (LTI) Systems

We will represent a system as follows, with $x(t)$ as an input into system $h(t)$ and $y(t)$ as the output. In next week's section, we will learn how to characterize a system in order to find $h(t)$, called the **impulse response**. This week, we will focus on how the properties of **linearity** and **time invariance** affect the inputs and outputs, $x(t)$ and $y(t)$ of a system.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Linearity

A **linear system** has the properties below:

(a) **additivity**

$$x_1(t)+x_2(t) \longrightarrow \boxed{h(t)} \longrightarrow y_1(t)+y_2(t) \tag{1}$$

(b) **scaling**

$$\alpha x(t) \longrightarrow \boxed{h(t)} \longrightarrow \alpha y(t) \tag{2}$$

Together, these two properties are known as **superposition**:

$$\alpha_1 x_1(t)+\alpha_2 x_2(t) \longrightarrow \boxed{h(t)} \longrightarrow \alpha_1 y_1(t)+\alpha_2 y_2(t)$$

Time invariance

A system is **time-invariant** if its behavior is fixed over time:

$$x(t-t_0) \longrightarrow \boxed{h(t)} \longrightarrow y(t-t_0) \tag{3}$$

Questions

1. Time-Shift Systems

Imagine we have a system $S_{\rightarrow 2}$ that takes any length 5 input signal and shifts it by 2 steps. For example, $S_{\rightarrow 2}([3,1,4,1,5]) = [1,5,3,1,4]$.

(a) Is this system linear? That is, for any signals $\vec{x}$ and $\vec{y}$, does $S_{\rightarrow 2}$ fulfill properties (1) and (2)?

(b) Is this system time-invariant? Does it fulfil (3)?

(c) What does $S_{\rightarrow 2}$ look like when written as a matrix?

2. Is it LTI?

Determine if the following systems are LTI:

(a) $y[t] = 2x[-2+3t] + 2x[2+3t]$

(b) $y[t] = 4^{x[t]}$

(c) $y[t] - y[t-1] + y[t-2] = x[t] - x[t-1] - x[t-2]$

(d) $y[t] = x[t] - x[t-2]$

(e) $y[t] = x[t] + tx[t-1]$

(f) $y[t] = 2^t cos(x[t])$

3. The DFT basis and LTI systems

Suppose $\vec{x}$ is the input signal applied to a linear time-invariant (LTI) system characterized by the impulse response $\vec{h}$. The output $\vec{y}$ is given by $C_{\vec{h}}\vec{x}$ where $C_{\vec{h}}$ is a circulant matrix with the first column given by $\vec{h}$. Suppose the DFT coefficients of $\vec{x}$ are given by

$$\vec{X} = \begin{bmatrix} X[0] & X[1] & \dots & X[n-1] \end{bmatrix}^T.$$

(a) Compute the DFT representation of the impulse response $\vec{h}$ as well as the eigenvalues of the circulant matrix that defines the system with that impulse response. ($n = 3$)

$$\vec{h} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

(b) Compute the DFT representation of the impulse response $\vec{h}$ as well as the eigenvalues of the circulant matrix that defines the system with that impulse response. ($n = 3$)

$$\vec{h} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) \end{bmatrix}^T$$

(c) Compute the DFT representation of the impulse response $\vec{h}$ as well as the eigenvalues of the circulant matrix that defines the system with that impulse response. ($n = N$)

$$\vec{h} = \begin{bmatrix} \cos\left(\frac{2\pi k}{N}(0)\right) & \cos\left(\frac{2\pi k}{N}(1)\right) & \cdots & \cos\left(\frac{2\pi k}{N}(n-1)\right) \end{bmatrix}^T$$

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