EECS 16B Designing Information Devices and Systems II
Spring 2017 Murat Arcak and Michel Maharbiz **Discussion 14B**

Notes

Recap: LTI System Model

Recall that $h[n]$, or the system response, is the system output to the input $\delta[n]$. We then decompose our input signal, $x[n]$, into a series of shifted δ functions.

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + \cdots + x[k-1]\delta[n-k+1] + \cdots$$

Since the system is linear and time invariant, the output is going to be,

$$y[n] = x[0]h[n] + x[1]h[n-1] + \cdots + x[k-1]h[n-k+1] + \cdots$$

If $x[n]$ and $h[n]$ are finite, we can construct a vector-matrix model of LTI systems. Let $x[n]$ have length P and $h[n]$ have length $Q+1$.

Important. When constructing the vector representations of $x[n]$ and $h[n]$, add zeros the end so that the length of both vectors are at least $P+Q$.

The vector-matrix model is as follows. Note the matrix structure.

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y[P+Q-1] \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} h[0] & 0 & 0 & \cdots & 0 & 0 & h[Q] & \cdots & h[1] \\ h[1] & h[0] & 0 & \cdots & 0 & 0 & 0 & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & 0 & 0 & 0 & \cdots & h[3] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h[Q-1] & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h[Q] & h[Q-1] & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & h[Q] & h[Q-1] & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & h[Q] & h[Q-1] & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & h[Q] & h[Q-1] & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & h[Q] & h[Q-1] & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & h[Q] & \cdots & \cdots & h[0] \end{bmatrix}}_{\boldsymbol{H}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[P-1] \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}} \tag{1}$$

Succinctly, we can represent this as,

$$\vec{y} = \boldsymbol{H}\vec{x}$$

DFT + LTI systems

Rather beautifully, DFT basis vectors are eigenvectors of $\boldsymbol{H}$. We will have $P+Q$ DFT vectors, since that is the dimensionality of our model.

$$u_k[n] = \frac{1}{\sqrt{P+Q}} e^{i\left(\frac{2\pi}{P+Q}k\right)n}$$

Let $\vec{u}_k$ be the vectorized version of $u_k[n]$ and let $H[k]$ be the k^{th} DFT coefficient of $h[n]$. (Recall that $H[k] = \langle \vec{h}, \vec{u}_k \rangle$).

$$\boldsymbol{H}\vec{u}_k = \underbrace{\left(\sqrt{P+Q} \times H[k]\right)}_{\text{eigenvalue}} \vec{u}_k$$

This is really useful. (1) can be really difficult to work with, so we simply decompose it into the DFT domain. Let Y be the DFT coefficient of $\vec{y}$, H be the DFT of $\vec{h}$ and X be the DFT of $\vec{x}$. Then,

$$\begin{bmatrix} Y[0] \\ Y[1] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ Y[P+Q-1] \end{bmatrix} = \sqrt{P+Q} \begin{bmatrix} H[0] & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & H[1] & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & H[2] & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & H[P+Q-1] \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ X[P+Q-1] \end{bmatrix} \tag{2}$$

In other words,

$$Y[k] = \sqrt{P+Q} H[k] X[k]$$

REMEMBER TO ZERO PAD WHEN VECTORIZING.

Questions

1. **Fast Convolution Using FFT**

   We have a system with finite impulse response $h(k)$, $k = 0, \ldots, M$.

   (a) How do you compute the response of this system to an input $u(k)$, $k = 0, \ldots, L-1$?

(b) How many operations (multiplications and additions) does this take?

(c) Can you think of a faster way to do this when $M+L$ is large? (Hint: computing the DFT using the Fast Fourier Transform (FFT) algorithm takes approximately $N \log N$ operations where N is the length of the vector being transformed.)

2. DFT and Bode Plots

Consider the circuit

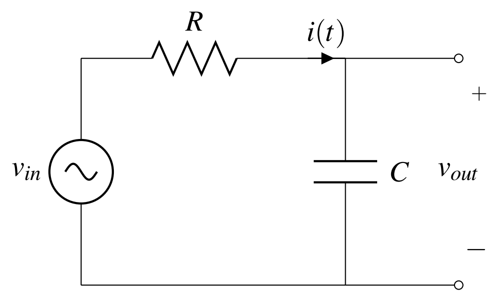

Figure 1: A first order RC Low Pass Filter

(a) Write a differential equation describing the evolution of v_{out}.

(b) Draw a Bode plot for this circuit.

(c) Discretize the differential equation assuming a zero order hold on the input v_{in}. (Hint: The differential equation $a\dot{x} = -x + u$ can be discretized as $x(k+1) = e^{-\frac{1}{a}T}x(k) + (1 - e^{-\frac{1}{a}T})u(k)$ where T is the sampling period.)

(d) What is the impulse response of this LTI system?

(e) In Python, compute the DFT of the impulse response. You can assume that $R = 1, C = 1/10, N = 1001$ (number of samples) and $T = 1/10$ (sampling period).

(f) Plot the amplitude and phase of the first $N/2$ DFT coefficients. How do they compare with the Bode plots?

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