Linearization of systems

One dimensional linear approximation

Consider a differentiable function f of one variable.

$$f : \mathbb{R} \to \mathbb{R}$$

Say we are interested in f in a small, open neighborhood about a particular point t_0. t_0 is often called the fixed point of the system. Lets call this open neighborhood U. In this case, we can construct a linear approximation of f about the neighborhood U. Recall that,

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t_0) \approx \frac{f(t) - f(t_0)}{t - t_0} \text{ for } t \in U$$

We can use the above to construct a linear approximation of f. Let f_l denote the linear approximation of f about U.

$$f_l(t) = \frac{\mathrm{d}f}{\mathrm{d}t}(t_0)(t - t_0) + f(t_0) \tag{1}$$

Strictly speaking, f_l is an affine approximation unless $t_0 = 0$, but the process of obtaining f_l is colloquially called the linearization of f.

For example, consider $f(t) = t^2$. We will set the fixed point of the system to be $t_0 = 1$. Then,

$$\begin{aligned} f_l(t) &= \tfrac{\mathrm{d}f}{\mathrm{d}t}(t_0)(t - t_0) + f(t_0) \\ &= 2(t-1) + 1 \\ &= 2t - 1 \end{aligned}$$

Let $\varepsilon = 10^{-2}$. Consider the open neighborhood $U = (1 - \varepsilon, 1 + \varepsilon)$. Lets plot f and f_l when their respective domains are restricted to U. This is seen in Figure 1.

Linearization of a system

Consider a continuous, non-linear system with state $\vec{x}(t)$ (which is n dimensional) and representation $\vec{u}(t)$ (which is m dimensional) of the form,

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t) = f(\vec{x}(t), \vec{u}(t))$$

To clarify and establish notation, f is a function that takes in $\vec{x}(t)$ and $\vec{u}(t)$ and outputs an n dimensional vector. $f_k(\vec{x}(t), \vec{u}(t))$ refers to the function that is the k^{th} coordinate of the output of $f(\vec{x}(t), \vec{u}(t))$.

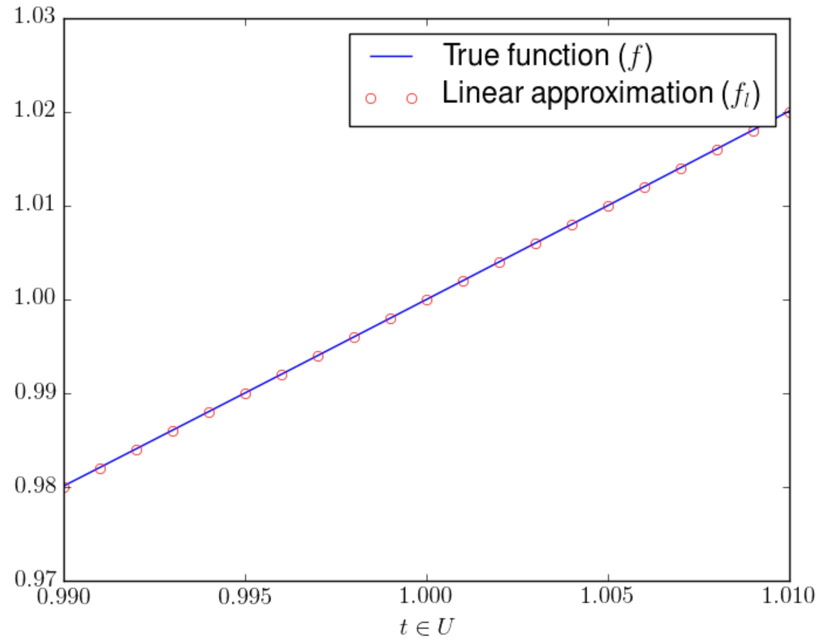

Figure 1: Comparing the linear approximation of a function to the original function

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Let $\vec{x}_0(t)$ be the desired state trajectory and $\vec{u}_0(t)$ be the desired input. Let $\delta\vec{x}(t)$ be a small perturbation about the state trajectory and $\delta\vec{u}(t)$ be the small perturbation about the input trajectory. In equation form,

$$x(t) = x_0(t) + \delta\vec{x}(t) \text{ and } u(t) = u_0(t) + \delta\vec{u}(t)$$

Analogous to (1), we will construct an affine estimate about a neighborhood around the fixed points.

$$f(\vec{x}_0(t) + \delta\vec{x}(t), \vec{u}_0(t) + \delta\vec{u}(t)) \approx f(\vec{x}_0(t), \vec{u}_0(t)) + \frac{\mathrm{d}f}{\mathrm{d}x}(x_0(t), u_0(t))\delta\vec{x}(t) + \frac{\mathrm{d}f}{\mathrm{d}u}(x_0, u_0)\delta\vec{u}(t) \tag{2}$$

Notice that we have used the notation $\frac{\mathrm{d}f}{\mathrm{d}x}$ and $\frac{\mathrm{d}f}{\mathrm{d}u}$, which might be strange to consider since $\vec{x}(t)$ is a vector and not a single variable. This is the multidimensional generalization of the derivative, which is constructed as follows.

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x,u) = \begin{bmatrix} \frac{\mathrm{d}f_1}{\mathrm{d}x_1}(x,u) & \frac{\mathrm{d}f_1}{\mathrm{d}x_2}(x,u) & \dots & \frac{\mathrm{d}f_1}{\mathrm{d}x_n}(x,u) \\ \frac{\mathrm{d}f_2}{\mathrm{d}x_1}(x,u) & \frac{\mathrm{d}f_2}{\mathrm{d}x_2}(x,u) & \dots & \frac{\mathrm{d}f_2}{\mathrm{d}x_n}(x,u) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\mathrm{d}f_n}{\mathrm{d}x_1}(x,u) & \frac{\mathrm{d}f_n}{\mathrm{d}x_2}(x,u) & \dots & \frac{\mathrm{d}f_n}{\mathrm{d}x_n}(x,u) \end{bmatrix} \text{ and } \frac{\mathrm{d}f}{\mathrm{d}u} = \begin{bmatrix} \frac{\mathrm{d}f_1}{\mathrm{d}u_1}(x,u) & \frac{\mathrm{d}f_1}{\mathrm{d}u_2}(x,u) & \dots & \frac{\mathrm{d}f_1}{\mathrm{d}u_m}(x,u) \\ \frac{\mathrm{d}f_2}{\mathrm{d}u_1}(x,u) & \frac{\mathrm{d}f_2}{\mathrm{d}u_2}(x,u) & \dots & \frac{\mathrm{d}f_2}{\mathrm{d}u_m}(x,u) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\mathrm{d}f_n}{\mathrm{d}u_1}(x,u) & \frac{\mathrm{d}f_2}{\mathrm{d}u_2}(x,u) & \dots & \frac{\mathrm{d}f_n}{\mathrm{d}u_m}(x,u) \end{bmatrix}$$

Note the dimensions of the matrices. It must be notated that, when calculating $\frac{\mathrm{d}f}{\mathrm{d}x}$, $\vec{u}$ is considered constant. Similarly, when calculating $\frac{\mathrm{d}f}{\mathrm{d}u}$, $\vec{x}$ is considered constant.

To continue from (2), note that,

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \frac{\mathrm{d}x_0(t)}{\mathrm{d}t} + \frac{\mathrm{d}\delta x}{\mathrm{d}t}(t) \text{ and } \frac{\mathrm{d}x_0}{\mathrm{d}t}(t) = f(x_0(t), u_0(t))$$

Plugging this back into (2), we get,

$$\cancel{\frac{\mathrm{d}x_0}{\mathrm{d}t}(t)} + \frac{\mathrm{d}\delta x}{\mathrm{d}t}(t) \approx \cancel{\frac{\mathrm{d}x_0}{\mathrm{d}t}(t)} + \frac{\mathrm{d}f}{\mathrm{d}x}(x_0(t), u_0(t))\delta\vec{x}(t) + \frac{\mathrm{d}f}{\mathrm{d}u}(x_0, u_0)\delta\vec{u}(t)$$

Thus, we get linearized version of our system.

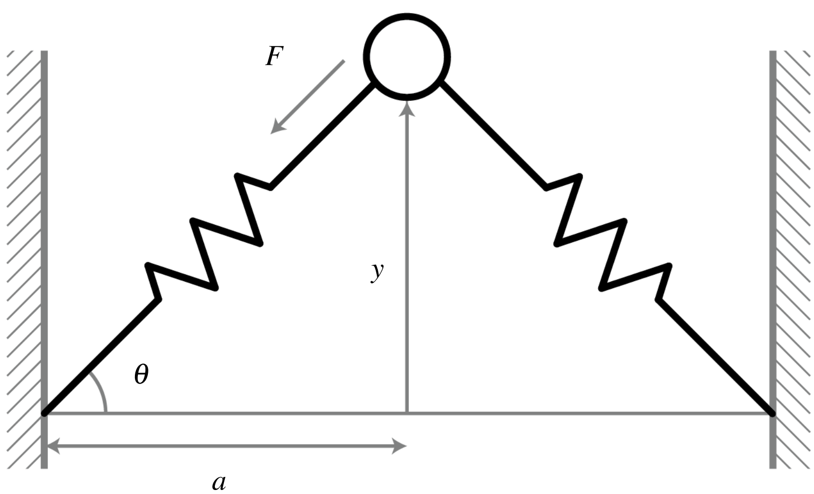
$$\frac{\mathrm{d}\delta x}{\mathrm{d}t}(t) \approx \frac{\mathrm{d}f}{\mathrm{d}x}(x_0(t), u_0(t))\delta\vec{x}(t) + \frac{\mathrm{d}f}{\mathrm{d}u}(x_0, u_0)\delta\vec{u}(t) \tag{3}$$

Note that, unlike the one dimensional linearization example, we are **linearizing with respect to $\vec{x}$ and $\vec{u}$**. Also, observe that our state variables are now the pertubations $\delta\vec{x}(t)$ and $\delta\vec{u}(t)$.

Questions

1. Linearization

Consider a mass attached to two springs:



We assume that each spring is linear with spring constant k and resting length X_0. We want to build a state space model that describes how the displacement y of the mass from the spring base evolves.

(a) Find the force F applied by each spring.

(b) Use Newton's law to write an equation for $\ddot{y}$ in terms of y.

(c) Write this model in state space form $\dot{x} = f(x)$.

(d) Find the equilibrium of the model assuming that $X_0 < a$.

(e) Linearize your model about the equilibrium.

(f) Compute the eigenvalues of your linearized model. Is this equilibrium stable?

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