Block Diagrams

Discrete Time

A useful way to visualize systems is to use block diagrams. We will introduce this topic via examples. Consider the following discrete system.

$$\begin{aligned} x_1[n+1] &= \quad x_1[n] - 2x_2[n] \\ x_2[n+1] &= -x_1[n] + 4x_2[n] + u[n] \end{aligned} \tag{1}$$

In matrix form, this can be represented as,

$$\vec{x}[n+1] = \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \vec{x}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[n]$$

We will know construct a block diagram representation. We will use a delay block to denote a delay of one. Refer to Figure **??**.

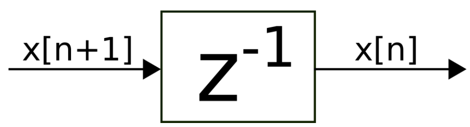


Figure 1: A simple block diagram denoting the delay block.

We will now construct a block diagram for the above system. This is in Figure **??**. Note the use of the adder blocks (the plus signs), the scaling blocks (the triangles) and the minus signs to indicate subtractions.

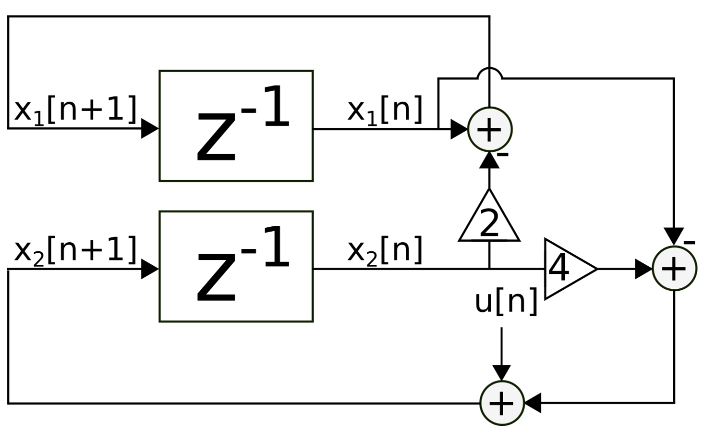


Figure 2: A block diagram denoting the system (**??**).

Continuous Time

It is a similar story in continuous time. The only difference is that instead of a delay block, we use an integrator block (denoted $\frac{1}{s}$ or $\int$). This is in Figure **??**.

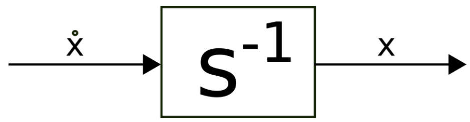


Figure 3: A block diagram denoting an integrator.

Observability

So far we have always assumed that we are able to directly see the A matrix, and we know the output of the system perfectly at all times. This isn't always the case, as the equations governing a given system are not always visible at the output (think of a robot car, we can see how the wheels move, but we are unable to see the voltages and currents in the motor actually governing the motion of the car).

$$\begin{aligned}\vec{x}(t+1) &= A\vec{x}(t) + B\vec{u}(t)\\ \vec{y}(t) &= C\vec{x}(t).\end{aligned}$$

In this system the $\vec{y}$ represents the outputs that we are able to see and measure when looking at the system. We are usually not able to directly observe the state space system of the matrix, or the contents of the A matrix. We define the concept of *Observability* as the ability to determine the initial state $\vec{x}(0)$ from a finite series of measurements. Using a similar argument as controllability, the observability matrix, O can be constructed the following way.

Constructing the Observability matrix

At time step zero we are able to see the output $\vec{y}(0) = C\vec{x}(0)$. If we let the system evolve with time without any control input, we can reconstruct the A matrix.

$$\begin{aligned}\vec{y}(0) &= C\vec{x}(0)\\ \vec{y}(1) &= C\vec{x}(1) = CA\vec{x}(0)\\ \vec{y}(2) &= CA^2\vec{x}(0)\\ &\vdots\\ \vec{y}(k) &= CA^k\vec{x}(0)\end{aligned}$$

Rewriting this, we get the following relation

$$\begin{bmatrix} \vec{y}(0) \\ \vec{y}(1) \\ \vec{y}(2) \\ \vec{y}(3) \\ \vdots \\ \vec{y}(k) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^k \end{bmatrix} \vec{x}(0)$$

If we want to be able to uniquely determine the original input $\vec{x}(0)$, we want each of the rows of this matrix to be linearly independent. By the Cayley-Hamilton theorem, we know that the maximum linearly independent power of A^kC is the one where $k = n-1$ where n is the dimension of our state space. Thus, we define the observability matrix O to be the following.

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

If the O matrix is full rank, then the system is observable, and we can uniquely determine the initial state from a series of inputs. Notice the similarity to the Controllability matrix. In fact, we call observability the mathematical dual of controllability.

1. Block Diagrams

(a) Create a block diagram for the system below:

$$\begin{bmatrix} s[n+1] \\ g[n+1] \\ r[n+1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s[n] \\ g[n] \\ r[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[n] + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} w[n]$$

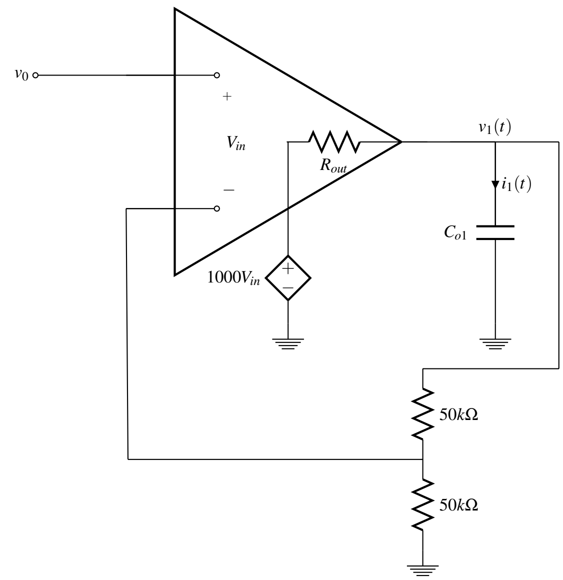
(b) Recall the linearized state space model for the ring oscillator below. Draw a block diagram describing this system.

$$\frac{d}{dt} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{10^{-8}} & 0 & -\frac{2}{10^{-8}} \\ -\frac{2}{10^{-8}} & -\frac{1}{10^{-8}} & 0 \\ 0 & -\frac{2}{10^{-8}} & -\frac{1}{10^{-8}} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix}$$

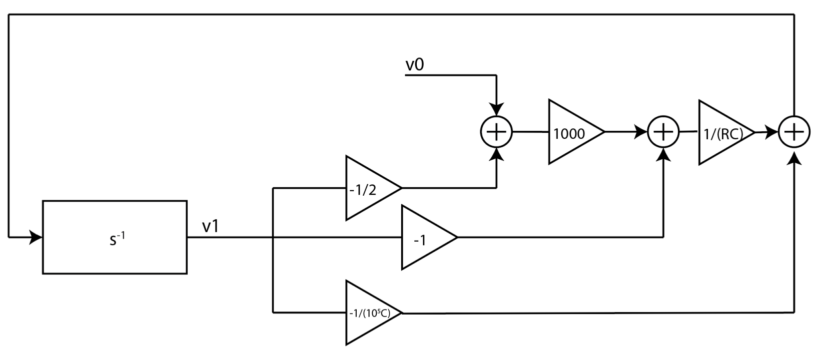
(c) Draw the block diagram for the closed-loop feedback system below:

$$\vec{x}[n+1] = \begin{bmatrix} 0 & 1 \\ a_1 + k_1 & a_2 + k_2 \end{bmatrix} \vec{x}[n]$$

(d) Draw the block diagram for the op-amp below:



$$\frac{dv_1}{dt} = \frac{1000(v_0 - \frac{1}{2}v_1) - v_1}{R_{out}C_{o1}} - \frac{v_1}{10^5 C_{o1}}$$



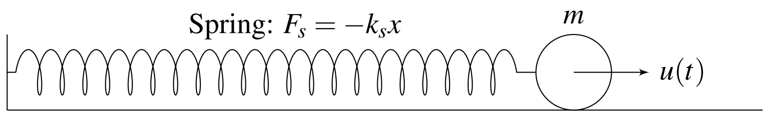
2. Observability

We would like to model a self-propelled mass connected by a string. It is subject to the following forces:

- The spring applies a force $F_s(t) = -k_s x(t)$ (at $x = 0$ it does not apply any force).
- The mass is propelled a force $u(t)$, the latest in perpertuum mobile technology. This is the only input of the system.

The only sensor the system has is a speedometer. That is, it can measure the speed of the system accurately, but it cannot directly measure its position.

Starting at an arbitrary position and speed, we would like to apply the correct inputs required to position the mass at $x = 0$.



(a) Model the system as a linear continuous-time state-space model:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + Bu(t)$$
$$y(t) = C \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

Write down A, B, and C.

(b) Given $m = 1$kg and $k_s = 2$. Plug the values into the system.

(c) Is the system observable?

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