EECS 16B Designing Information Devices and Systems II
Spring 2017 Murat Arcak and Michel Maharbiz Discussion 8A

Controller Canonical Form

When working with systems in state-space, you may have noticed that a single system can be represented in many different forms, depending on factors such as how you ordered your state vector. Writing out systems in certain **canonical forms** often allows engineers to quickly determine system behavior.

The **controller canonical form**, which guarantees controllability and simplifies eigenvalue placement, takes on the following form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Change of basis to controller canonical

Given a controllable system of the form $\frac{d}{dt}\vec{x}(t) = \tilde{A}\vec{x}(t) + \tilde{B}u(t)$, we can transform it into controller canonical form by choosing some T such that:

$$\vec{z} = T\vec{x}, A = T\tilde{A}T^{-1}, \text{ and } T\tilde{B} = B$$

for matrices A and B of the form shown in the previous section.

We can calculate this T using $\boxed{T = C\tilde{C}^{-1}}$, where C an $\tilde{C}$ are the controllability matrices of the controller canonical and original forms, respectively.

Notice that when we place our system in feedback using $u(t) = K\vec{z} = \tilde{K}\vec{x}$ with $\tilde{K} = KT$, we get the closed loop matrix

$$T(\tilde{A} + \tilde{B}\tilde{K})T^{-1} = A + BK$$

The eigenvalues of both systems are the same, and we can manipulate eigenvalues as we please using methods we have previously learned.

Coefficient match to controller canonical

For smaller systems, we can use a technique called **coefficient matching** to bring the system to controller canonical form. Different forms of a system are similarity transforms of each other. This means that the eigenvalues of a system are preserved. Thus, we can find the characteristic polynomial of our system in its original form, and then map the coefficients of this polynomial into the state matrix of our system in controller canonical form.

Let's walk through an example to convert

$$\frac{d}{dt}\vec{x} = \tilde{A}\vec{x} + \tilde{B}u = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$

to controller canonical form

$$\frac{d}{dt}\vec{z} = A\vec{z} + Bu = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}\vec{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

Our first step should always be to **check for controllability** to ensure that a transformation into controllable canonical form actually exists.

$$\tilde{C} = \begin{bmatrix} \tilde{A}\tilde{B} & \tilde{B} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Now that we have verified controllability, we can calculate the characteristic polynomial of $\tilde{A}$.

$$det(\tilde{A} - \lambda I) = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1$$

The characteristic polynomial of A is

$$det(A - \lambda I) = -\lambda(a_2 - \lambda) - a_1 = \lambda^2 - \lambda a_2 - a_1$$

From here we can see that $a_2 = 2$ and $a_1 = -1$, giving us the controller canonical form

$$\frac{d}{dt}\vec{x} = A\vec{x} + Bu = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

Observers

So far, we have neglected to mention how our state-feedback derive the information about the state, when the state is not directly measurable.

We need to construct an additional system that provides an estimate $\vec{\hat{x}}$ of the state $\vec{x}$ using the inputs $\vec{u}$ and the outputs $\vec{y}$. Given the system

$$\begin{aligned} \vec{x}(t+1) &= A\vec{x}(t) + B\vec{u}(t) \\ \vec{y}(t) &= C\vec{x}(t)\,. \end{aligned}$$

We construct an additional system, called an *observer*, to estimate the state:

$$\vec{\hat{x}}(t+1) = A\vec{\hat{x}}(t) + B\vec{u}(t) - \underbrace{L\left(\vec{\hat{y}}(t) - \vec{y}(t)\right)}_{\text{output feedback}}$$

$$\vec{\hat{y}}(t) = C\vec{\hat{x}}(t)\,.$$

The additional output-feedback term tracks the difference between the estimated and real outputs. If the observer outputs deviate from the real outputs, it nudges the observer closer to the real outputs. If L is chosen the right way, the estimate quickly converges to the real state.

Subtracting the system from its observer, we can find a difference equation describing the error $\vec{e}(t) = \vec{\hat{x}}(t) - \vec{x}(t)$:

$$\vec{e}(t+1) = (A - LC)\vec{e}(t)$$

This equation is similar to state feedback, but the order of LC vs. BK is different. To handle this difficulty we introduce the **dual system**. It is a formal system—we do not care about its behavior—we just use it to find L.

$$\vec{z}(t+1) = A^T\vec{z}(t) + C^T\vec{v}(t)$$

This system is controllable if the original system is observable (the controllability matrix of the dual system is the transpose of the observability matrix of the original system).

We now find the feedback L^T that places the eigenvalues of $A^T - C^T L^T$ where we want them to be. Then transpose the result and get $A - LC$ with the the same eigenvalues. By setting $\vec{v}(t) = L^T\vec{z}(t)$, we can put this system into feedback and set the eigenvalues as we please.

Finally, we can now write down the closed loop system with feedback and observer. The input is $\vec{u}(t) = K\vec{\hat{x}}(t) = K(\vec{x}(t) + \vec{e}(t))$:

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t) = A\vec{x}(t) + BK(\vec{x}(t) + \vec{e}(t)) = (A + BK)\vec{x}(t) + BK\vec{e}(t)$$

Resulting in

$$\begin{bmatrix} \vec{x}(t+1) \\ \vec{e}(t+1) \end{bmatrix} = \begin{bmatrix} A + BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ \vec{e}(t) \end{bmatrix}.$$

In continuous time the math is the same, except for (1) we replace the $\vec{x}(t+1)$ by $\frac{d}{dt}\vec{x}(t)$, and (2) we place the eigenvalues so their real part is less than 0.

1. Control Canonical Form - Introduction

(a) Show that for any square matrix A and any invertible matrix T, the matrix TAT^{-1} has the same characteristic polynomial as A.

(b) Compute the characteristic polynomial of

$$A = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{bmatrix}$$

(c) Put the system

$$\dot{x}(t) = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

in control canonical form without computing the transformation matrix T.

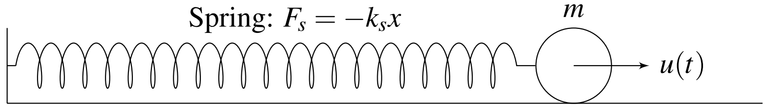
2. Observers

We would like to model a self-propelled mass connected by a string. It is subject to the following forces:

- The spring applies a force $F_s(t) = -k_s x(t)$ (at $x = 0$ it does not apply any force).
- The mass is propelled a force $u(t)$, the latest in perpertuum mobile technology. This is the only input of the system.

The only sensor the system has is a speedometer. That is, it can measure the speed of the system accurately, but it cannot directly measure its position.

Starting at an arbitrary position and speed, we would like to apply the correct inputs required to position the mass at $x = 0$.



From the last discussion, we have the following model for the spring system given given $m = 1$kg and $k_s = 2$.

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

Recall that we determined it was observable.

(a) Write down the equations of the observer system.

(b) Write down the equations governing the estimation error:
$$\vec{e}\,(t) = \begin{bmatrix} e_0 \\ e_1 \end{bmatrix}.$$

(c) Compute an output-feedback L that places both the eigenvalues of the system governing the estimation error at -2.

(d) Write out the dual system of the original system.

(e) Is the dual system controllable?

(f) Write out the observer closed loop-system using L.

(g) Is the original system stable?

(h) Is the original system controllable?

(i) Derive a state-feedback F that places both the eigenvalues of the closed-loop system at -1.

(j) Write down the equations for the closed loop system, including the feedback and observer.

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