Singular Value Decomposition

The SVD is a useful way to characterize a matrix. Let A be a matrix from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or $A \in \mathbb{R}^{m \times n}$). The SVD of matrix A is the following decomposition into 3 different matrices that have specific properties discussed below.

$$A = U\Sigma V^T$$

The properties of U, Σ and V are,

- U is an $[m \times m]$ matrix whose columns consist of orthonormal vectors that form a basis for the codomain, or $\mathbb{R}^m$. Consequently,
$$U^*U = I$$
- V is an $[n \times n]$ matrix whose columns consist of orthonormal vectors that form a basis for the domain, or $\mathbb{R}^n$. Consequently,
$$V^*V = I$$
- U characterizes the column space of A and V characterizes the row space of A.
- Σ is an $[m \times n]$ matrix whose diagonal entries are the singular values of A arranged in descending order. The singular values are the square roots of the eigenvalues of A^TA (or, identically, AA^T).
- The singular values of a matrix A are always real and non-negative.

We calculate the SVD as follows.

(a) Pick AA^T or A^TA.

(b) Let λ_i denote the i^{th} eigenvalue of the chosen matrix, Λ be the diagonal matrix consisting of the eigenvalues of the chosen matrix and σ_i be the i^{th} singular value of A.

- If using A^TA, calculate its eigenvalue decomposition to find Σ and V.
$$A^*A = V\Lambda V^* \text{ with } \sigma_i = \sqrt{\lambda_i}$$
- If using AA^T, calculate its eigenvalue decomposition to assign Σ and U
$$AA^T = U\Lambda U^T \text{ with } \sigma_i = \sqrt{\lambda_i}$$

Note that the non-diagonal entries of Σ will be zero. The various singular values $\{\sigma_i\}$ calculated will be on the diagonal.

(c) Regardless of which matrix picked in step (1), calculate the remaining matrix (either U or V) using the following to ensure everything is sign consistent.

$$Av_i = \sigma_i u_i$$

If any singular value is 0 or you seem to have run out of vectors to completely construct the U or V matrix, complete the basis (or columns of the appropriate matrix) using Gram-Schmidt. Remember to orthonormalize afterwards.

The SVD of A is taken to better understand it in terms of the 3 nice matrices. Often, we do not completely construct the U and V matrices.

Questions

1. SVD and Fundamental Subspaces

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

(a) Find the SVD of A.

(b) Find the rank of A.

(c) Find a basis for the kernel (or nullspace) of A.

(d) Find a basis for the range (or columnspace) of A.

2. Eigenvalue Decomposition and Singular Value Decomposition

Consider a matrix $A \in \mathbb{R}^{n \times n}$ with the eigenvalue decomposition,

$$A = P \Lambda P^T \text{ with } \lambda_i \geq 0$$

(a) Find the SVD of A.

(b) Let one particular eigenvalue λ_j be negative, with the associated eigenvector being p_j. Succinctly,

$$Ap_j = \lambda_j p_j \text{ with } \lambda_j < 0$$

We are still assuming that,

$$A = P \Lambda P^T$$

i. What is the singular value σ_i associated to λ_i?

ii. What is the relationship between the left singular vector u_i, the right singular vector v_i and the eigenvector p_i?

3. SVD and Induced 2-Norm

(a) Show that if U is an orthogonal matrix then for any $\vec{x}$

$$\|U\vec{x}\| = \|\vec{x}\|.$$

(b) Find the maximum

$$\max_{\{\vec{x}:\|\vec{x}\|=1\}} \|A\vec{x}\|$$

in terms of the singular values of A.

(c) Find the $\vec{x}$ that maximizes the expression above.

Contributors:

- Siddharth Iyer.
- John Maidens.