1. MIMO wireless signals

Ever wonder why newer wifi routers and cellular base stations have 4 or sometimes even more antennas on them? New wireless technologies actually use multiple antennas that each send their own signal on the same frequency band. The key here is not only do we encode signals in frequency bands, but also in spatial ones using a technique known as ‘Spatial Multiplexing’.

We call this idea ‘MIMO’ wireless, which stands for "multiple input multiple output". This technique is used in many standards including 802.11n/ac, 4G LTE, and WiMAX.

In this problem, we will explore how signals are decoded on the output end.

Consider the following:

We have 2 transmit antennas and 3 receive antennas, each receive antenna gets some signal from each of the receive antennas. We can model the input output relation of the system as follows:

\[
\begin{bmatrix}
    h_{11} & h_{21} \\
    h_{12} & h_{22} \\
    h_{13} & h_{23}
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t)
\end{bmatrix}
= \begin{bmatrix}
    y_1(t) \\
    y_2(t) \\
    y_3(t)
\end{bmatrix}
\]

or

\[H\tilde{x}(t) = \tilde{y}(t)\]

Here, \(H\) is the spatial-response matrix and it acts on the signals instantaneously at each time. For the purpose of this problem, we are going to pretend that there are no echoes across time.
(a) With our new MIMO wireless system, we want to recover the original $\vec{x}(t)$ signal after receiving the $\vec{y}(t)$. In order to do this, we will left multiply $\vec{y}(t)$ by some matrix $A$; ideally we should then exactly recover $\vec{x}(t)$ ($A\vec{y}(t) = \vec{x}(t)$). Using the SVD to decompose $H$, analytically write down what this matrix $A$ should be.

(b) How is the solution you found in part (a) related to the least squares solution of this problem?

(c) What we just did is referred to as "post processing" or "post coding", and involves the receive end having more antennas than the send side. Many times this is not the case (e.g. a wireless cell tower having many more antennas than a phone). What if we wanted to send 2 streams on 3 antennas and receive precisely those 2 streams back on the other end?

The channel is very similar to the one we had made in part (a). In fact, the original channel modelled with spatial response matrix $H$ is precisely the transpose of this channel! Thus, we can say the spatial response matrix for this channel, lets call it $H'$ is simply the following:

$$H' = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{bmatrix} = H^T$$

Using the SVD of $H$ and its relation to $H'$, show how you can pre-process $x_1(t)$ and $x_2(t)$ so that you recover them precisely after they have been transmitted across the channel. To be more explicit, after the processing and transmission has been done $y_1(t) = x_1(t), y_2(t) = x_2(t)$.

(d) (Optional) Why do you think that we are using the SVD here? Is there a unique solution of what to transmit that will achieve the desired goal in the previous part? Why choose this approach?

2. Sinc functions

The sinc function is defined as,

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

(a) Verify that,

$$\lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} = 1$$

*Hint. Look into L’Hospital’s Rule.*
(b) Verify that,
\[ \frac{1}{\pi} \int_{0}^{\pi} \cos(\omega x) d\omega = \text{sinc}(x) \]

(c) If we think of the above integral as a sum of oscillatory functions of different frequencies, the \text{sinc} function has continuous frequency components in the range \([0, \pi]\). What is the range of frequencies of the function,
\[ f(x) = \text{sinc}\left(\frac{x}{T}\right) \]?

*Hint.* Try to use substitution and the above integral.

(d) Continuing the idea of integration as the continuous version of sums, we define the continuous inner product between two functions as,
\[ \langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x) dx \]

Given that the length squared of a vector \(\vec{v}\) of length \(n\) is,
\[ \|\vec{v}\|^2 = \langle v, v \rangle = \sum_{i=1}^{n} v_i \overline{v_i}, \]
what is the analogous length squared of a function, \(f\), using the inner product definition above? This is a one-line answer.

We will only consider functions where \(\|f\| < \infty\), so the inner product is well defined. It turns out that the set of shifted sinc functions are orthonormal with the above inner product! Let,
\[ \phi_k(x) = \text{sinc}(x-k) \text{ where } k \text{ is an integer}. \]

Then,
\[ \|\phi_k\| = 1 \text{ and } \langle \phi_m, \phi_n \rangle = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases} \]

3. **The vector space of polynomials**

A polynomial of degree at most \(n\) on a single variable can be written as
\[ p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n \]
where we assume that the coefficients \(p_0, p_1, \ldots, p_n\) are real. Let \(P_n\) be the vector space of all polynomials of degree at most \(n\).

(a) Consider the representation of \(p \in P_n\) as the vector of its coefficients in \(\mathbb{R}^{n+1}\).
\[ \vec{p} = \begin{bmatrix} p_0 & p_1 & \cdots & p_n \end{bmatrix}^T \]

Show that the set \(\mathcal{B}_n = \{1, x, x^2, \ldots, x^n\}\) forms a basis of \(P_n\), by showing the following.

- Every element of \(P_n\) can be expressed as a linear combination of elements in \(\mathcal{B}_n\).
- No element in \(\mathcal{B}_n\) can be expressed as a linear combination of the other elements of \(\mathcal{B}_n\).

*(Hint: Use the aspect of the fundamental theorem of algebra which says that a nonzero polynomial of degree \(n\) has at most \(n\) roots, and use a proof by contradiction.)*
(b) Suppose that the coefficients \( p_0, \ldots, p_n \) of \( p \) are unknown. To determine the coefficients, we evaluate \( p \) on \( n + 1 \) points, \( x_0, \ldots, x_n \). Suppose that \( p(x_i) = y_i \) for \( 0 \leq i \leq n \). Find a matrix \( V \) in terms of the \( x_i \), such that

\[
V \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.
\]

(c) For the case where \( n = 2 \), compute the determinant of \( V \) and show that it is equal to

\[
\det(V) = \prod_{0 \leq i < j \leq n} (x_j - x_i).
\]

Conclude that if \( x_0, \ldots, x_n \) are distinct, then we can uniquely recover the coefficients \( p_0, \ldots, p_n \) of \( p \). This holds for \( n > 2 \) in general, but consider only the case where \( n = 2 \) for now.

(d) (Optional) Argue using Lagrange interpolation that indeed such matrices \( V \) above must always be invertible if the \( x_i \) are distinct.

(e) We can define an inner product on \( P_n \) by setting

\[
\langle p, q \rangle = \int_{-1}^{1} p(x)q(x) \, dx.
\]

Show that this satisfies the following properties of a real inner product. (We would have to put in a complex conjugate on \( p \) if we wanted a complex inner product.)

- \( \langle p, p \rangle \geq 0 \), with equality if and only if \( p = 0 \).
- For all \( a \in \mathbb{R} \), \( \langle ap, q \rangle = a \langle p, q \rangle \).
- \( \langle p, q \rangle = \langle q, p \rangle \).

(f) Now that we have an inner product on \( P_n \), we can consider orthonormality. If \( \mathcal{B} = \{ b_0, b_1, \ldots, b_n \} \) is a basis for \( P_n \), we say that it is an orthonormal basis if

- \( \langle b_i, b_j \rangle = 0 \) if \( i \neq j \).
- \( \langle b_i, b_i \rangle = 1 \).

We can also compute projections. For any \( p, u \in P_n, u \neq 0 \), the projection of \( p \) onto \( u \) is

\[
\text{proj}_u p = \frac{\langle p, u \rangle}{\langle u, u \rangle} u.
\]

Consider the case where \( n = 2 \). From part (a), we have the basis \( \{ 1, x, x^2 \} \) for \( P_2 \). Convert this into an orthonormal basis using the Gram-Schmidt process.

(g) (Optional) An alternative inner-product could be placed upon real polynomials if we simply represented them by a sequence of their evaluations at \( 0, 1, \ldots, n \) and adopted the standard Euclidean inner product on sequences of real numbers. Can you give an example of an orthonormal basis with this alternative inner product?

4. Lagrange interpolation by polynomials

Given \( n \) distinct points and the corresponding evaluations/sampling of a function \( f(x) \), \( (x_i, f(x_i)) \) for \( 0 \leq i \leq n - 1 \), the Lagrange interpolating polynomial is the polynomial of the least degree which passes through all the given points.
Given \( n \) distinct points and the corresponding evaluations, \((x_i, f(x_i))\) for \( 0 \leq i \leq n - 1 \), the Lagrange polynomial is

\[
P_n(x) = \sum_{i=0}^{n-1} f(x_i)L_i(x),
\]

where

\[
L_i(x) = \prod_{j=0, j \neq i}^{n-1} \frac{x - x_j}{x_i - x_j} = \frac{(x - x_0)}{(x_i - x_0)} \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \cdots \frac{(x - x_n - 1)}{(x_i - x_{n-1})}.
\]

Here is an example: for two data points, \((x_0, f(x_0)) = (0, 4), (x_1, f(x_1)) = (-1, -3)\), we have

\[
L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - (-1)}{0 - (-1)} = x + 1
\]

and

\[
L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{(-1) - (0)} = -x
\]

Then

\[
P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) = 4(x + 1) + (-3)(-x) = 7x + 4
\]

We can sketch those equations on the 2D plane as follows:

(a) Given three data points, \((2, 3), (0, -1)\) and \((-1, -6)\), find a polynomial \( f(x) = ax^2 + bx + c \) fitting the three points. Do this by solving a system of linear equations for the unknowns \( a, b, c \). Is this polynomial unique?

(b) Like the monomial basis \( \{1, x, x^2, x^3, \ldots\} \), the set \( \{L_i(x)\} \) is a new basis for the subspace of degree \( n \) or lower polynomials. \( P_n(x) \) is the sum of the scaled basis polynomials. Find the \( L_i(x) \) corresponding to the three sample points in (a). Show your steps.
(c) Find the Lagrange polynomial $P_n(x)$ for the three points in (a). Compare the result to the answer in (a). Are they different from each other? Why or why not?

(d) Sketch $P_n(x)$ and each $f(x_i)L_i(x)$ on the 2D plane.

(e) Show that the Lagrange interpolating polynomial must pass through all given points. In other words, show that $P_n(x_i) = f(x_i)$ for all $x_i$. Do this in general, not just for the example above.

Contributors:

- Saavan Patel.
- Siddharth Iyer.