1. Interpolation

Samples from the sinusoid \( f(x) = \sin(0.2\pi x) \) are shown in Figure 1. Draw the results of interpolation using each of the following three methods:

(a) Zero order hold interpolation.
(b) Linear interpolation.
(c) Sinc interpolation assuming the Nyquist limit has been satisfied.

![Figure 1: Samples of f(x).](image)

2. Linear interpolation

Consider a piecewise-linear real valued function \( f(x) \) such that,

(a) In the interval \([k, k + 1]\) where \( k \) is an integer, \( f \) is a line straight line.
(b) \( f(x) \) is zero for \( x < k_1 \) and \( f(k_1) = 0 \).
(c) \( f(x) \) is zero for \( x > k_2 \) and \( f(k_2) = 0 \).

Consider the function \( \phi(x) \) defined as,

\[
\phi(x) = \begin{cases} 
1 - |x|, & x \in [-1, 1] \\
0, & \text{otherwise}
\end{cases}
\]

(a) Sketch \( \phi(x-k) \) for some arbitrary integer \( k \).
(b) Write the basis function and coefficient that captures the line of \( f(x) \) from \( x = k_1 \) to \( x = k_1 + 1 \). That is to say, find real number \( \alpha \) and integer \( p \) such that,

\[
f(x) = \alpha \phi(x - p) \text{ for } x \in [k_1, k_1 + 1]
\]

(c) What is the equation of the line from \( k \) to \( k + 1 \), where \([k, k + 1]\) is within \([k_1, k_2]\)? That is to say, find an equation of the form,

\[
y = mx + c,
\]

that represents the line in \( f \) between \( k \) and \( k + 1 \).

(d) Consider the function,

\[
g(x) = f(k)\phi(x - k) + f(k + 1)\phi(x - (k + 1))
\]

What is the equation of the line formed between \([k, k + 1]\), where \([k, k + 1]\) is within \([k_1, k_2]\)? Write it once again in the form,

\[
y = mx + c.
\]

This should match your previous answer.

(e) Given the answers to the previous parts, we have shown that we can break down \( f \) into a linear sum of shift \( \phi \) functions. Find the coefficients \( \alpha_k \) such that,

\[
f(x) = \sum_{k \in \mathbb{Z}} \alpha_k \phi(x - k)
\]

3. **Sampling a continuous-time control system to get a discrete-time control system**

Recall from Lecture 12A that a continuous-time system

\[
\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \tag{1}
\]

\[
y(t) = C\vec{x}(t)
\]

can be represented with a discrete-time model

\[
\vec{x}_d(k + 1) = A_d\vec{x}(k) \tag{2}
\]

\[
y_d(k) = C\vec{x}_d(k)
\]

where \( \vec{x}_d(k) \) and \( y_d(k) \) are the values of the state \( \vec{x}(t) \) and output \( y(t) \) at time instants \( t = kT, k = 1, 2, 3, \ldots \)

In this problem we will see that the observability of (1) does not necessarily imply observability of (2): there may be sampling periods \( T \) that fail to preserve observability. Since observability depends on \( A \) and \( C \) alone we have omitted the inputs in the equations above.

(a) Suppose \( A \) is diagonalizable; that is, there exists a matrix \( P \) such that

\[
P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}
\]
Show that
\[ A_d = P \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix} P^{-1}. \]

To do so, you can introduce the new state vector \( \vec{z} = P^{-1} \vec{x} \) and then use the result from Lecture 12A for the discretization of a diagonal \( A \) matrix to obtain a discrete-time model for \( \vec{z}_d(k) \). You would then return to the original state with \( \vec{x}_d(k) = P\vec{z}_d(k) \) to obtain (2).

(b) Use the result of part (a) to calculate \( A_d \) as a function of the sampling period \( T \) when
\[ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \]

(c) Let \( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \) so that \((A, C)\) is an observable pair. Show that there exist values of \( T \) for which \((A_d, C)\) is not observable. (Hint: compare the discrete-time model to the example in Lecture 7B.)

4. Aliasing

Watch the following video: [https://www.youtube.com/watch?v=jQDjJRyMeWg](https://www.youtube.com/watch?v=jQDjJRyMeWg).

Assume the video camera running at 30 frames per second. That is to say, the camera takes 30 photos within a second, with the time between photos being constant.

(a) Given that the main rotor has 5 blades, list all the possible rates at which the main rotor is spinning in revolutions per second assuming no physical limitations. 

*Hint: Your answer should depend on \( k \) where \( k \) can be any integer.*

(b) Given that the back rotor has 3 blades and completes 2 revolutions in 1 second in the video, list all the possible rates at which the back rotor is spinning in revolutions per second assuming no physical limitations.

*Hint: Your answer should depend on \( k \) where \( k \) can be any integer.*

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