EECS 16B Designing Information Devices and Systems II
Spring 2017 Murat Arcak and Michel Maharbiz **Homework 13**

This homework is optional but in scope.

1. Conjugate symmetry

The DFT basis is naturally complex. However, many signals that we are interested in understanding are real-valued. It is natural to wonder if anything special happens to real-vectors viewed in the DFT basis.

Why are such questions natural? Because a complex number can be viewed as a pair of real numbers. So, when we take n real numbers and transform them into n complex numbers, it is natural to wonder where the extra n real numbers have come from? Here we will see how the DFT exhibits something that we can call "conjugate symmetry" and that really, there are only n real numbers that determine everything.

(a) Convert $\vec{h} = [1,2,1,0]^T$ to the DFT basis by computing $U^*\vec{h}$, where U is the DFT basis with $n = 4$. Plot the components using both Cartesian and Polar coordinates. Do you see something?

(b) Do the same for $\vec{h} = [2,1,-1]^T$ by computing $U^*\vec{h}$, where U is now the DFT basis with $n = 3$. Do you see something?

(c) Let $\vec{x}$ be a real vector of length n, and let $\vec{X} = U^*\vec{x}$ be $\vec{x}$ in the DFT basis. Show that the k-th component of $\vec{X}$ satisfies $X[k] = (X[n-k])^*$, for $0 \le k \le \lfloor \frac{n}{2} \rfloor$. Check that this holds in parts (a) and (b).
thus we could conclude $X[k] = (X[n-k])^*$, for $0 \le k \le \lfloor \frac{n}{2} \rfloor$.

(d) Show that $X[0]$ is real if $\vec{x}$ is real. Check that this holds in parts (a) and (b).

(e) If n is even, show that $X[\frac{n}{2}]$ is real if $\vec{x}$ is real. Check that this holds in part (a).

2. Sampling and the DFT

Recap of DFT: Consider a continuous-time signal $x(t)$. We can collect a vector of discrete samples of $x(t)$ over time as $\vec{x}$, which is a finite time signal of length n.

$$\vec{x} = \begin{bmatrix} x[0] & \dots & x[n-1] \end{bmatrix}^T \tag{1}$$

As we learned from lecture, this signal can be represented in the DFT basis. Let $\vec{X} = \begin{bmatrix} X[0] & \dots & X[n-1] \end{bmatrix}^T$ be the coordinates of $\vec{x}$ in the DFT basis.

$$\vec{X} = U^{-1}\vec{x} = U^*\vec{x} \tag{2}$$

where U is a matrix of the DFT basis vectors.

$$U = \begin{bmatrix} | & & | \\ \vec{u}_0 & \cdots & \vec{u}_{n-1} \\ | & & | \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{\frac{2\pi i}{n}} & e^{\frac{2\pi i(2)}{n}} & \cdots & e^{\frac{2\pi i(n-1)}{n}} \\ 1 & e^{\frac{2\pi i(2)}{n}} & e^{\frac{2\pi i(4)}{n}} & \cdots & e^{\frac{2\pi i2(n-1)}{n}} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{\frac{2\pi i(n-1)}{n}} & e^{\frac{2\pi i2(n-1)}{n}} & \cdots & e^{\frac{2\pi i(n-1)(n-1)}{n}} \end{bmatrix} \tag{3}$$

Recall that U is an orthonormal basis, and the above equation shows that $\vec{x} = U\vec{X}$. That is, $\vec{x}$ can be always represented as a linear combination of the DFT basis signals $\vec{u}_m$ with coefficients $X[m]$.

$$\vec{x} = X[0]\vec{u}_0 + \cdots + X[n-1]\vec{u}_{n-1} \tag{4}$$

Applying U to $\vec{X}$ is a process called "taking the inverse DFT," which is the reconstruction of the original signal as a superposition of its sinusoidal components.

(a) The n-th roots of unity are the roots of the equation

$$x^n = 1,$$

Write down all complex roots of the above equation. Make sure they satisfy this equation. The roots should be in this form: $x = e^{i\square}$

(b) Assume $n = 4$. Label all the 4-th roots of unity on the complex plane. Call them s_0, s_1, s_2, s_3.

(c) One perspective on the DFT basis vectors is that these are obtained by looking at the powers of each of the roots-of-unity. Verify that the j-th DFT basis element is just $\frac{1}{\sqrt{n}}(s_j)^k$ as $k = 0, \ldots, n-1$.

(d) An alternative perspective on the DFT treats the $s_0, \ldots, s_{n-1}$ as places where we can evaluate polynomials. Verify that the j-th DFT basis element is just $\frac{1}{\sqrt{n}}r^j$ evaluated as r goes through the n s_k points, as $k = 0, \ldots, n-1$.

(e) Consider the complex signal $\vec{x}$ below,

$$\vec{x} = \begin{bmatrix} 2 \\ 1 + \frac{\sqrt{3}i}{2} + \frac{1}{2} \\ x[2] \\ 0 \\ x[4] \\ x[5] \end{bmatrix}, \tag{5}$$

along with its corresponding DFT coefficients, $\vec{X}$. Assume that we know $X[m] = 0$, for all $m \geq 3$. Because we have 3 samples and exactly 3 unknowns, we are able to recover all entries of $\vec{x}$ based on the above information. Do so and write down all entries of $\vec{x}$ and $\vec{X}$ explicitly.

(f) Consider a length n discrete-time signal $\vec{x}$. Suppose that we know that its DFT coefficients $\vec{X}$ satisfy $X[m] = 0$, for all $m \geq k$. What is the minimum number of sampling points we need to interpolate a unique $\vec{x}$?

3. DFT of a sinusoid

Let

$$x(t) = \sin(0.4\pi t), \quad t = 0, 1, \ldots, N-1 \tag{6}$$

with $N = 100$.

(a) Find the DFT coefficients $X(k), k = 0, \ldots, 99$.

(b) Show that the coefficients in part (a) satisfy the conjugate symmetry property.

(c) Now let $N = 101$. Using the command `numpy.fft.fft(x, norm="ortho")` find the real and imaginary parts of $X(k), k = 0, \ldots, 100$, and plot them against the frequency variable $\omega \in [0, 2\pi]$. (Recall that the integer k corresponds to the frequency $\frac{2\pi}{N}k$.) You should see that the DFT plot peaks around the frequency of (6). What is the interpretation of the second peak?

4. Two-dimensional DFT

The concept of DFT can be extended to 2-dimensional signals. For example, images are considered as 2D signals in the "image domain." We can model a 2D signal as an $n \times n$ array M_f where

$$M_f = \begin{bmatrix} M_f[0,0] & M_f[0,1] & \cdots & M_f[0,n-1] \\ M_f[1,0] & M_f[1,1] & \cdots & M_f[1,n-1] \\ \vdots & \vdots & \cdots & \vdots \\ M_f[n-1,0] & M_f[n-1,1] & \cdots & M_f[n-1,n-1] \end{bmatrix} \tag{7}$$

Let the matrix M_F be the 2D DFT coefficients of M_f. The 2D DFT coefficients are given by

$$M_F[k,l] = \frac{1}{n}\sum_{u=0}^{n-1}\sum_{v=0}^{n-1} M_f[u,v]e^{-i\frac{2\pi}{n}(ku+lv)} \tag{8}$$

Conversely, an image-domain signal can be expressed by the linear combinations of its 2D DFT coefficients.

$$M_f[u,v] = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} M_F[k,l]e^{i\frac{2\pi}{n}(ku+lv)} \tag{9}$$

In this problem, we will play with images with 2D DFT using the IPython notebook. Use the provided IPython notebook file accompanying this homework.

(a) The equation (8) is the result of doing DFT on the column vectors of M_f first (let the result be M_f'), and then applying DFT again to the row vectors of M_f'. Demonstrate this using the IPython notebook with the image `campus_256.jpg`.

(b) The equation (9) is the result of doing inverse DFT on the row vectors of M_F first (let the result be M_F'), and then applying inverse DFT again to the column vectors of M_F'. Demonstrate this fact using the IPython notebook with the image `campus_256.jpg`.

(c) We can modify the 2D DFT coefficients of an image by zeroing out its high- or low-frequency coefficients. Use the IPython notebook to demonstrate the effects of modifying the 2D DFT coefficients of the image `campus_256.jpg`.

(d) Look at the provided image `einstein_monroe_256.jpg` and comment about what is special of this image. Use the IPython notebook to separate the hybrid image into one with a low-frequency Marilyn Monroe and one with high-frequency Albert Einstein.

5. LTI filter

Suppose we apply the length $L = 100$ input

$$u(t) = \cos(0.1\pi t) + \cos(0.4\pi t), \quad t = 0, 1, \ldots, 99 \tag{10}$$

to a finite impulse response filter whose impulse response is

$$h(t) = \begin{cases} \frac{1}{6} & t = 0, \ldots, 5 \\ 0 & \text{otherwise.} \end{cases}$$

We want to find the output $y(t)$, $t = 0, \ldots, 104$ from the DFT.

(a) Find the 105-point DFT of $u(t)$ by adding 5 zeros to the length-100 signal given above and using the DFT command `numpy.fft.fft(x, norm="ortho")`. Plot the magnitude $|U(k)|$ against the frequency variable $\omega \in [0, 2\pi]$. (Recall that the integer k corresponds to the frequency $\frac{2\pi}{105}k$.)

(b) Find the 105-point DFT of $h(t)$ by adding 99 zeros to the length-6 impulse response given above. Plot the magnitude $|H(k)|$ against the frequency variable $\omega \in [0, 2\pi]$. Given this frequency response how do you think each frequency component of $u(t)$ will be affected when the filter is applied?

(c) As will be shown in Lecture 14A, the DFT coefficients of the output are

$$Y(k) = \sqrt{105}H(k)U(k),\ k = 0, \ldots 104.$$

Find $y(t)$ using the inverse DFT command `numpy.fft.ifft(x, norm="ortho")` Plot both $u(t)$ and $y(t)$ versus time t and explain how the filter modified $u(t)$.

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