This homework is due February 15, 2017, at 17:00. This homework is OPTIONAL. However, you are highly encouraged to do this homework since everything on it, except for bode plots, is within the scope of midterm 1.

1. Transfer functions

Consider the circuit below.

The circuit has an input phasor voltage \( \tilde{V}_i \) at frequency \( \omega \) rad/sec applied at the input terminals shown in the illustration above, causing an output phasor voltage \( \tilde{V}_o \) at output terminals.

(a) Obtain an expression for \( H = \frac{\tilde{V}_o}{\tilde{V}_i} \) in terms of \( Z_{R1}, Z_{R2}, Z_{C1}, Z_{C2} \) (they are the impedance of \( R_1, R_2, C_1, C_2 \), respectively).

**Solution:** Define the voltage phasor \( \tilde{V}_1 \) as shown below.

From KCL, we have

\[
\frac{\tilde{V}_1 - \tilde{V}_i}{Z_{R1}} + \frac{\tilde{V}_1}{Z_{C1}} + \frac{\tilde{V}_1}{Z_{R2} + Z_{C2}} = 0
\]  
(1)

\[
\tilde{V}_i = \tilde{V}_1
\]  
(2)

\[
\left( \frac{1}{Z_{R1}} + \frac{1}{Z_{C1}} + \frac{1}{Z_{R2} + Z_{C2}} \right) Z_{R1} \tilde{V}_1 = \tilde{V}_1
\]  
(3)

We also know that \( \tilde{V}_1 \) and \( \tilde{V}_o \) are related by (voltage divider)

\[
\tilde{V}_o = \tilde{V}_1 \frac{Z_{C2}}{Z_{R2} + Z_{C2}}
\]  
(4)
Combining the above formulas, we have
\[ V_o = \tilde{V}_i \frac{Z_{C_2}}{Z_{R_2} + Z_{C_2}} \left( \frac{1}{\frac{1}{Z_{R_1}} + \frac{1}{Z_{C_1}} + \frac{1}{Z_{R_2} + Z_{C_2}}} \right) Z_{R_1} Z_{C_2} \]
(5)

\[ = \frac{\tilde{V}_i Z_{C_1} Z_{C_2}}{Z_{R_1} Z_{C_1} + Z_{C_1} Z_{C_2} + Z_{R_1} Z_{C_2} + Z_{R_2} Z_{C_1} + Z_{R_1} Z_{R_2}} \]
(6)

Thus,
\[ H = \frac{V_o}{V_i} = \frac{Z_{C_1} Z_{C_2}}{Z_{R_1} Z_{C_1} + Z_{C_1} Z_{C_2} + Z_{R_1} Z_{C_2} + Z_{R_2} Z_{C_1} + Z_{R_1} Z_{R_2}} \]
(7)

(b) Obtain an expression for \( H(\omega) = \tilde{V}_o/\tilde{V}_i \) in the form of
\[ H(\omega) = \frac{1}{1 + j2\xi(\omega/\omega_c) + (j\omega/\omega_c)^2}, \]
given that \( R_1 = 1\Omega, R_2 = 2\Omega, C_1 = 1F, \) and \( C_2 = 2F \). What are the values of \( \xi \) and \( \omega_c \)?

**Solution:** The impedance in the phasor domain is given by
\[ Z_{R_1} = R_1 \quad Z_{R_2} = R_2 \quad Z_{C_1} = \frac{1}{j\omega C_1} \quad Z_{C_2} = \frac{1}{j\omega C_2} \]

Using the result in part (a), we have
\[ H(\omega) = \frac{\frac{1}{j\omega C_1} \frac{1}{j\omega C_2}}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} + \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} + \frac{1}{j\omega C_1}} \frac{1}{1 + j\omega(\frac{R_1}{C_1} + (\frac{R_1}{C_1} + \frac{R_2}{C_2})C_2) + (j\omega)^2\frac{R_1}{C_1} + \frac{R_2}{C_2}C_1C_2} \]
(8)

\[ = \frac{1}{1 + j\omega(7) + (j\omega)^2(4)} \]
(9)

\[ = \frac{1}{1 + j\frac{27}{4}(\omega/(1/2)) + (j\omega/(1/2))^2} \]
(10)

By inspection, we have \( \xi = 7/4 \) and \( \omega_c = 1/2 \).

(c) We can express the transfer function \( H(\omega) \) in the polar form. That is,
\[ H(\omega) = M(\omega)e^{j\phi(\omega)} \]
The functions \( M(\omega) \) and \( \phi(\omega) \) are the magnitude and the phase angle of \( H(\omega) \), respectively. Write down \( M(\omega) \) and \( \phi(\omega) \) using the transfer function you derived in part (b).

**Solution:** Rewriting the result in part (b), we have
\[ H(\omega) = \frac{1}{1 - 4\omega^2 + j7\omega} \]
(12)

Taking the magnitude, we have
\[ M(\omega) = \frac{1}{\sqrt{(1 - 4\omega^2)^2 + (7\omega)^2}} \]
(13)
For the phase angle, be careful using the $\tan^{-1}(\cdot)$ function, which is only correct for complex numbers in the first or the fourth quadrants of the complex plane. For complex numbers in the second and the third quadrants, we need to shift their $\tan^{-1}(\cdot)$ by $\pi$.

$$\phi(\omega) = \begin{cases} 
-\tan^{-1}\left(\frac{7\omega}{1-4\omega^2}\right) & \text{when } 0 \leq \omega \leq \frac{1}{2} \\
-\tan^{-1}\left(\frac{7\omega}{1-4\omega^2}\right) - \pi & \text{when } \omega > \frac{1}{2} 
\end{cases}$$ (14)

Note that $1 - 4\omega^2 + j7\omega$ is in the first quadrant when $\omega < 1/2$ and in the second quadrant when $\omega > 1/2$.

(d) Compute the phasors of $H(0)$, $H(\omega_c)$, and $H(\infty)$ using the results in part (b) and (c).

**Solution:**

$$H(0) = M(0)e^{j\phi(0)} = 1e^{j0} = 1$$ (15)

$$H(\infty) = M(\infty)e^{j\phi(\infty)} = 0e^{-j\pi} = 0$$ (16)

$$H(\omega_c) = H(1/2) = \frac{2}{7}e^{-j\frac{3\pi}{4}}$$ (17)

(e) Consider the circuit below.

The voltage source is given by

$$v_i(t) = 12\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$$

The values of $R_1$, $R_2$, $C_1$, and $C_2$ are the ones given in part (b). Obtain an expression for $v_o(t)$ in the form of $\alpha\cos\left(\frac{1}{2}t + \theta\right)$.

**Solution:** The phasor of $v_i(t)$ is $\tilde{V}_i = 12e^{-j\frac{3\pi}{4}}$. The phasor of $v_o(t)$ is (note that $\omega_c = 1/2$)

$$\tilde{V}_o = H(\omega)\tilde{V}_i = H(1/2)\tilde{V}_i = \frac{2}{7}e^{-j\frac{3\pi}{4}}12e^{-j\frac{3\pi}{4}} = \frac{24}{7}e^{-j\frac{3\pi}{4}}$$ (18)

Thus,

$$v_o(t) = \Re\{\tilde{V}_o e^{j\omega t}\} = \Re\{\frac{24}{7}e^{-j\frac{3\pi}{4}}e^{j\frac{5\pi}{4}t}\} = \frac{24}{7}\cos\left(\frac{1}{2}t - \frac{5\pi}{4}\right)$$ (19)

2. Active filter

(a) problem 9.44 from the textbook
Problem 9.44  Show that the transfer function of the circuit shown in Fig. P9.44 is given by

\[ H(\omega) = \frac{V_o}{V_s} = -G \left(1 + j\frac{\omega}{\omega_c}\right), \]

and relate \( G \) and \( \omega_c \) to \( R_1, R_2, \) and \( C. \)

Figure P9.44  Circuit for Problem 9.44.

Figure 1: Reproduced from Ulaby, Maharbiz, Furse. Circuits. Third Edition. with permission

Solution:
3. Bass-booster

RC circuits and filters are very useful for altering the frequency content of signals. For example, audio equalization equipment can use these filters to adjust the pitch content of audio signals. Suppose we want to boost the bass of our favorite music, we can use the active filter circuit below to tune the frequency content of our favorite jams.

**Problem 9.44** Show that the transfer function of the circuit shown in Fig. P9.44 is given by

\[
H(\omega) = \frac{V_0}{V_s} = -G \left(1 + j\frac{\omega}{\omega_c}\right),
\]

and relate \(G\) and \(\omega_c\) to \(R_1\), \(R_2\), and \(C\).

![Figure P9.44](image)

**Solution:**

\[
H(\omega) = \frac{V_0}{V_s} = -\frac{-R_2}{R_1 \parallel (-j\omega C)}
= -\frac{R_2}{\frac{jR_1}{\omega C}} \times \left(R_1 - \frac{j}{\omega C}\right)
= -\frac{R_2}{R_1} (1 + j\omega R_1 C)
= -G \left(1 + j\frac{\omega}{\omega_c}\right)
\]

with

\[
G = \frac{R_2}{R_1}, \quad \omega_c = \frac{1}{R_1 C}.
\]
Figure 2: Audio Equalizer Circuit

$C_f = 400 \, \text{nF}$, $R_f = 1 \, \text{k}\Omega$, $R_g = 100 \, \text{Ω}$, and $R_s = 1 \, \text{k}\Omega$. $R_1$ and $R_2$ are both variable capacitors that can be used to tune the frequency balance of our output signal. Assume the input audio signal is $V_{in} = \cos(\omega t)$.

(a) First treat the first two active filters as disconnected and independent from the third (see figure below). Find expressions for $H_1(\omega) = \frac{\tilde{V}_1}{\tilde{V}_{in}}$ and $H_2(\omega) = \frac{\tilde{V}_2}{\tilde{V}_{in}}$, where $\tilde{V}_{in}$ and $\tilde{V}_1$, and $\tilde{V}_2$ are the phasor transformations of the voltages labeled on the circuit. Sketch the bode plots for the magnitude of these transfer functions. What kind of filter is each transfer function?
**Solution:** First we start with the current entering the RC network in the top filter. We can find this because we know that the (-) terminal of the op-amp is at ground.

\[
I = \frac{\tilde{V}_{\text{in}}}{R_s}
\]

Then we find the impedance of the RC network:

\[
Z_1 = \frac{R_f}{1 + j\omega 400 \times 10^{-6}}
\]

Combining these two, we get:

\[
H_1(\omega) = \frac{\tilde{V}_1}{\tilde{V}_{\text{in}}} = -\frac{1}{1 + j\omega 400 \times 10^{-6}}
\]

This is a lowpass filter.

Following a similar process, for the second RC network we get:

\[
I = \frac{\tilde{V}_{\text{in}}}{R_s + \frac{1}{j\omega C_f}}
\]

\[
\tilde{V}_2 = -\frac{\tilde{V}_{\text{in}} j\omega C_f R_f}{1 + j\omega R_s C_f}
\]
\[ H_2(\omega) = \frac{V_2}{V_{in}} = -\frac{j\omega400 \times 10^{-6}}{1 + j\omega400 \times 10^{-6}} \]

This is a high pass filter.

(b) If we connect the outputs, \( \tilde{V}_1 \) and \( \tilde{V}_2 \), to \( R_1 \) and \( R_2 \), respectively, will \( \tilde{V}_1 \) and \( \tilde{V}_2 \) change? Can we use our expressions for \( \tilde{V}_1 \) and \( \tilde{V}_2 \) that we found in the previous part to represent \( \tilde{V}_1 \) and \( \tilde{V}_2 \) in the full three op amp circuit?

**Solution:** No, these outputs won’t change. Since the outputs are driven by an op amp, they won’t be affected by any loads connected. Therefore, we can use these expressions to represent \( \tilde{V}_1 \) and \( \tilde{V}_2 \) of the full circuit.

(c) Find a function that describes \( \tilde{V}_{out} \) in terms of \( \tilde{V}_1 \) and \( \tilde{V}_2 \)

**Solution:** This part of the circuit is an op amp summer, so the currents of the two inputs are summed before entering \( R_g \). The resulting equation is

\[ \tilde{V}_{out} = -R_g \left( \frac{\tilde{V}_1}{R_1} + \frac{\tilde{V}_2}{R_2} \right) \]

\[ \tilde{V}_{out} = -100 \left( \frac{\tilde{V}_1}{R_1} + \frac{\tilde{V}_2}{R_2} \right) \]

(d) Combine the results of the last two parts to find an overall transfer function for \( H_{ov}(\omega) = \frac{\tilde{V}_{out}}{V_{in}} \).

**Solution:**

\[ H_{ov}(\omega) = \frac{R_g}{R_1} \frac{1}{1 + j\omega R_f C_f} + \frac{R_g}{R_2} \frac{j\omega R_s C_f}{1 + j\omega R_s C_f} \]

\[ H_{ov}(\omega) = \frac{100}{R_1} \frac{1}{1 + j\omega400 \times 10^{-6}} + \frac{100}{R_2} \frac{j\omega400 \times 10^{-6}}{1 + j\omega \times 400 \times 10^{-6}} \]

(e) Using this circuit, how could we set \( R_1 \) and \( R_2 \) to boost our bass frequency signals (\( f < 400 \text{ Hz} \)) without affecting mid and treble range signals? Sketch a bode plot of the magnitude of your overall bass-boosting function.

**Solution:** We can increase \( R_2 \) while keeping \( R_1 \) the same to increase bass signals. An example bode plot is below with \( R_1 = 50\Omega \) and \( R_2 = 100\Omega \). Your bode plot doesn’t need to look exactly like this, but it should look similar.

![Bode Plot](image)
Contributors:

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