1. LLR circuit

Consider the circuit in Figure 1 driven by a voltage source with voltage $u(t)$. The output $y(t)$ is the current through the resistor and the state variables are the inductor currents as marked in the circuit diagram.

![LLR circuit diagram](image)

**Figure 1: LLR circuit**

(a) Write a state model for this circuit.

**Solution:** From KCL we know $y = x_1 - x_2$. Now using KVL around the main cycle we get

\[
-u + L_1 \frac{dx_1}{dt} + Ry = 0
\]

\[
\Rightarrow \frac{dx_1}{dt} = -\frac{R}{L_1} y + \frac{1}{L_1} u
\]

\[
\Rightarrow \frac{dx_1}{dt} = -\frac{R}{L_1} x_1 + \frac{R}{L_1} x_2 + \frac{1}{L_1} u
\]

Now using KVL around the second subcycle we get

\[
L_2 \frac{dx_2}{dt} - Ry = 0
\]

\[
\Rightarrow \frac{dx_2}{dt} = \frac{R}{L_2} y
\]

\[
\Rightarrow \frac{dx_2}{dt} = \frac{R}{L_2} x_1 - \frac{R}{L_2} x_2
\]
So the equations can be written in state space form as

\[
\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R L_1}{L_2} & \frac{R}{L_2} \\ \frac{R}{L_2} & -\frac{R L_2}{L_1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\]

(b) Find all equilibrium points when \( u(t) = 0 \) for all \( t \).

**Solution:** Setting \[
\begin{bmatrix} -\frac{R L_1}{L_2} & \frac{R}{L_2} \\ \frac{R}{L_2} & -\frac{R L_2}{L_1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 0
\]
we see that there is an entire subspace of equilibrium points \( x_1 = x_2 \).

(c) Determine if the system is controllable.

**Solution:** The controllability matrix

\[
\mathcal{C} = [AB] = \begin{bmatrix} -\frac{R}{L_1} & \frac{1}{L_1} \\ \frac{R L_1}{L_2} & 0 \end{bmatrix}
\]

has full rank. So the system is controllable.

(d) Determine if the system is observable.

**Solution:** The observability matrix

\[
\mathcal{O} = [CA] = \begin{bmatrix} -\left(\frac{R L_1}{L_2} + \frac{R}{L_2}\right) \\ \left(\frac{R}{L_1} + \frac{R}{L_2}\right) \end{bmatrix}
\]

has rank 1. So the system is not observable.

(e) If your answer to part (c) or (d) is no, explain the physical reason for lack of controllability or observability, whichever is applicable.

**Solution:** Note that at any of the equilibria in part (b) the output is identically zero. So there is no way to distinguish between for example the system being in configuration \( x_1 = 1, x_2 = 1 \) from \( x_1 = 0, x_2 = 0 \).

2. **Inverted pendulum**

Consider the inverted pendulum depicted below, whose equations of motion are

\[
\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2 \theta} \left( \frac{u}{m} + \dot{\theta}^2 \ell \sin \theta - g \sin \theta \cos \theta \right)
\]

\[
\dot{\theta} = \frac{1}{\ell \left(\frac{M}{m} + \sin^2 \theta\right)} \left( -\frac{u}{m} \cos \theta - \dot{\theta}^2 \ell \cos \theta \sin \theta + \frac{M + m}{m} g \sin \theta \right).
\]
Consider the inverted pendulum system depicted below.

To bring \( \vec{x}(t) \) to the equilibrium \( \vec{x} = 0 \) we apply
\[
\vec{u}(t) = K \vec{x}(t)
\]
and obtain the closed-loop system
\[
y(t), \quad \text{leading to a fourth order model.}
\]
We now design a state feedback controller,
\[
\vec{u}(t) = k_1 \theta(t) + k_2 \dot{\theta}(t) + k_3 \dot{y}(t).
\]

(a) Write the state model using the variables \( x_1(t) = \theta(t), \) \( x_2(t) = \dot{\theta}(t), \) and \( x_3(t) = \dot{y}(t). \) We do not include \( y(t) \) as a state variable because we are interested in stabilizing the point \( \theta = 0, \dot{\theta} = 0, \dot{y} = 0, \) and we are not concerned about the final value of the position \( y(t) \).

**Solution:** We have,
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \left( \frac{1}{l(\frac{m}{m} + \sin^2(x_1))} \right) \left( -\frac{m}{m} \cos(x_1) - x_2^2 l \cos(x_1) \sin(x_1) + \frac{M+m}{m} g \sin(x_1) \right) \\
\dot{x}_3 &= \left( \frac{1}{\frac{m}{m} + \sin^2(x_1)} \right) \left( \frac{u}{m} + x_2^2 l \sin(x_1) - g \sin(x_1) \cos(x_1) \right)
\end{align*}
\]
\( \triangleq f_1(x_1, x_2, x_3, u) \)
\( \triangleq f_2(x_1, x_2, x_3, u) \)
\( \triangleq f_3(x_1, x_2, x_3, u) \)

(b) Linearize this model at the equilibrium \( x_1 = 0, \) \( x_2 = 0, \) \( x_3 = 0, \) and indicate the resulting \( A \) and \( B \) matrices.

**Solution:** We can keep in mind that \( x_1 = x_2 = x_3 = 0 \) to make the derivative much easier. Since we aren’t asked to linearize about a particular input, we can linearize about \( u^* = 0. \) This is fine because \( f_2 \) and \( f_3 \) are affine (linear plus a constant term) with respect to \( u. \)

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1}(0,0,0,0) &= 0 \\
\frac{\partial f_1}{\partial x_2}(0,0,0,0) &= 1 \\
\frac{\partial f_1}{\partial x_3}(0,0,0,0) &= 0 \\
\frac{\partial f_2}{\partial x_1}(0,0,0,0) &= \frac{M+m}{M} g \\
\frac{\partial f_2}{\partial x_2}(0,0,0,0) &= 0 \\
\frac{\partial f_2}{\partial x_3}(0,0,0,0) &= 0 \\
\frac{\partial f_3}{\partial x_1}(0,0,0,0) &= -\frac{m}{M} g \\
\frac{\partial f_3}{\partial x_2}(0,0,0,0) &= 0 \\
\frac{\partial f_3}{\partial x_3}(0,0,0,0) &= 0
\end{align*}
\]

And,
\[
\begin{align*}
\frac{\partial f_1}{\partial u}(0,0,0,0) &= 0 \\
\frac{\partial f_2}{\partial u}(0,0,0,0) &= -\frac{1}{M} \\
\frac{\partial f_3}{\partial u}(0,0,0,0) &= \frac{1}{M}
\end{align*}
\]

Since \( x^* = 0 \) and \( u^* = 0, \) we can use the same state variables \( x \) and \( u. \) Then,
\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{M} g & 0 & 0 \\ -\frac{m}{M} g & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{M} \\ \frac{1}{M} \end{bmatrix} u
\]

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(c) Show that the linearized model is controllable.

**Solution:** Observe that,

\[
AB = \begin{bmatrix}
- \frac{1}{lM} & 0 & 0
\end{bmatrix}^T
\]

and,

\[
A^2B = \begin{bmatrix} 0, & -\frac{M+m}{(lM)^2g}, & \frac{m}{lM^2g} \end{bmatrix}^T
\]

Then,

\[
C = \begin{bmatrix}
0 & -\frac{1}{lM} & 0 & \frac{M+m}{(lM)^2g} & \frac{m}{lM^2g}
\end{bmatrix}
\]

Since we are trying to test rank, we can remove scalar terms from the vectors. We then get,

\[
\text{rank } C = \text{rank } \begin{bmatrix}
0 & 1 & 0 & \frac{M+m}{(lM)^2g} & \frac{m}{lM^2g}
\end{bmatrix}
\]

We want to show that,

\[
\frac{m}{M+m} \neq 1
\]

This must be the case because this is only true when \( M = 0 \), which is not possible since an object at this macro scale must have mass.

(d) Suppose \( M = 1 \), \( m = 0.1 \), \( l = 1 \), and \( g = 10 \), and design a state feedback controller,

\[
u(t) = k_1 \theta(t) + k_2 \dot{\theta}(t) + k_3 \ddot{y}(t),
\]

such that the eigenvalues of \( A + BK \) (the “closed-loop eigenvalues”) are \( \lambda_1 = \lambda_2 = \lambda_3 = -1 \).

**Solution:** Plugging in values, the system is,

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} u
\]

Setting \( u = K \ddot{x} \), we get,

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

or,

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 11 - k_1 & -k_2 & 0 \\ k_1 - 1 & -k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]
The characteristic polynomial is,
\[ p(A+BK)(\lambda) = \lambda^3 + \lambda^2(k_2 - k_3) + \lambda(k_1 - 11) + 10k_3 = 0 \]

Our target polynomial is,
\[ p(A+BK)(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 \]

Comparing coefficients, we get,
\[ k_1 = 14, k_2 = 3.1, k_3 = 0.1 \]

(e) Suppose we set \( k_2 = k_3 = 0 \) and vary only \( k_1 \); that is, the controller uses only \( \theta(t) \) for feedback. Does there exist a \( k_1 \) value such that all closed-loop eigenvalues have negative real parts?

**Solution:** The characteristic polynomial is,
\[ p(A+BK)(\lambda) = \lambda^3 + \lambda(k_1 - 11) = 0 \]

No matter what \( k_1 \) is, there will always be an eigenvalue at 0.

3. **Continuous-time analog observer design: ship autopilots**

Modern ships use autopilots for steering. The main task of the autopilot is to maintain constant heading. A common system model used for ship steering controllers is the Nomoto first-order model. It is described using the following differential equation:
\[ T \ddot{\psi} + \dot{\psi} = K \delta, \]

where \( \psi \) is the ship heading, \( \delta \) is the rudder angle, and \( K \) and \( T \) are constants that are empirically estimated during sea trials. The “dot” notation used here is the physics convention (Newton’s notation) that is very convenient for problems where nothing more that a second derivative is needed.

The only sensor is a gyrocompass, which reports the ship’s current heading \( y(t) = \psi(t) \). We would also like to provide a good estimate of an additional important parameter, the rate of turn — the derivative of the ship’s current heading.

The input of the ship model is the rudder angle \( \delta \), and the output is the heading \( \psi \), as measured by the gyrocompass.

(Note for the curious: undoubtably, some of you are wondering why we don’t just take the derivative of the measurement and be done with it. The reason is that although we are describing everything without any noise, in the real-world, all measurements are noisy. Taking the derivative of noise is a very bad idea because it is in the nature of noise to shake a lot and so the derivative gets swamped by the shaking of the noise.)

In this problem you’ll construct an analog continuous-time observer, and then analyze its behaviour.

(a) Choose your state variables so that you have a two-dimensional state.

**Solution:** This is a second-order differential equation in terms of \( \psi \). We can introduce a new variable \( r = \dot{\psi} \), the rate of turn. Together with \( \psi \), our state would be \( \begin{bmatrix} \psi \\ r \end{bmatrix} \).
(b) Write down the system as a state-space model with a two-dimensional state.

**Solution:** Replacing $\dot{\psi}$ with $r$ and rearranging the terms, the equation becomes:

$$T\dot{r} = K\delta - r.$$  

And in matrix form:

$$\frac{d}{dt} \begin{bmatrix} \psi(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \psi(t) \\ r(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} \delta(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi(t) \\ r(t) \end{bmatrix}$$

c) Is the system observable?

**Solution:** Yes.

$$CA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The observability matrix has two linearly independent vectors.

d) Write down a model for the observer in matrix form using $\ell$ to represent how you weigh the difference between the observed output $y(t)$ and the estimated output $\hat{y}(t)$ coming from within your observer.

**Solution:**

$$\frac{d}{dt} \begin{bmatrix} \hat{\psi}(t) \\ \hat{r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix} \begin{bmatrix} \hat{\psi}(t) \\ \hat{r}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K}{T} \end{bmatrix} \delta(t) - \ell (\hat{y}(t) - y(t))$$

$$\hat{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\psi}(t) \\ \hat{r}(t) \end{bmatrix}$$

Note that the first row of the equation is

$$\dot{\hat{\psi}} = \hat{r} - l_0 (\hat{y} - y)$$

We also know that $\dot{\psi} = r$. Does that mean we can cancel out $\dot{\psi}$ and $\hat{r}$?

The answer is no. $\hat{r}$ does not necessarily equal $\dot{\psi}$. These are just estimates of the original quantities, we intentionally add a new term $-l_0 (\hat{y} - y)$ to make sure the two are not always equal, but rather converge into the original $\psi$ and $r$ in time.

e) Find $\ell = \begin{bmatrix} l_0 \\ l_1 \end{bmatrix}$ to place both the eigenvalues of the estimation error evolution at $-2$.

**Solution:**

$$A - \ell C = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix} - \begin{bmatrix} l_0 \\ l_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_0 & 1 \\ -l_1 & \frac{1}{T} \end{bmatrix}$$
The characteristic polynomial is:

\[ \lambda^2 + \left( l_0 + \frac{1}{T} \right) \lambda + \left( \frac{l_0}{T} + l_1 \right) \]

To place the eigenvalues at -2, the characteristic polynomial should be \((\lambda + 2)^2 = \lambda^2 + 4\lambda + 4\). Therefore

\[
\begin{bmatrix}
l_0 \\
l_1
\end{bmatrix} = \begin{bmatrix}
4 - \frac{1}{T} \\
2 - \frac{1}{T}
\end{bmatrix}
\]

Now that we have designed the output-feedback and placed the eigenvalues of the estimation error. We’ll design a circuit implementing the observer.

We will represent the state variables as voltages. Each input, output, and state variable will be implemented as a node in our circuit. The output of the original systems (the gyrocompass) would be an input of this system, and so would the rudder angle.

Recall that in EE16A and previously in EE16B, you have seen how to implement the following operations using simpler circuit elements (mainly resistors, capacitors and op-amps): differentiation, integration, scaling, addition and negation. This will be enough to implement the observer.

(f) Design a circuit whose output is the integral of its input with respect to time.

**Solution:** We build an inverting integrator with an inverting amplifier connected in series:

![Circuit Diagram](image)

This circuit’s output is

\[ v_{out} = \frac{R_3}{R_1 R_2 C} \int_0^t v_{in}(u) du. \]

We need to choose resistor and capacitor values so that \( R_1 R_2 C = R_3 \). For example \( R_1 = R_2 = 1\,M\Omega \), \( C = 1nF \), \( R_3 = 1k\Omega \).

(g) Design a circuit whose output is a scaled version by a constant \( a_0 \) of its input.

**Solution:** We can just use an non-inverting amplifier
This circuit’s output is

\[ v_{\text{out}} = v_{\text{in}} \frac{R_1 + R_2}{R_1} \]

Choose resistor values such that \( \frac{R_1 + R_2}{R_1} = a_0 \). If the constant is negative, we can always connect an inverting amplifier with a gain of -1.

(h) Design a circuit whose output is the negation of its input.

**Solution:** We can just use an inverting amplifier

This circuit’s output is

\[ v_{\text{out}} = -v_{\text{in}} \frac{R_2}{R_1} \]

Choose resistor values to set the right gain so \( R_2 = R_1 \) which can be 100\,\Omega.

(i) Design a circuit whose output is the sum of its two inputs.

**Solution:** We can just use an inverting summing amplifier. In this example we have three inputs:
This circuit’s output is
\[ v_{\text{out}} = \frac{R_4 R_2}{R_1 R_3} (v_{\text{in1}} + v_{\text{in2}}) \]

We need to choose resistor values such that \( R_2 R_4 = R_1 R_3 \) which can be done by say, setting them all to 100kΩ.

Now that we have the basic circuit elements. We’ll implement the observer as a circuit.

(j) Use the circuits you designed above to construct the observer as a circuit driven by the output of the gyrocompass.

Solution: We can rearrange the system into the following form:

\[
\begin{bmatrix}
\dot{\psi} \\
\dot{\hat{r}}
\end{bmatrix} =
\begin{bmatrix}
f(\hat{\psi}, \hat{\hat{r}}, \delta, y) \\
g(\hat{\psi}, \hat{\hat{r}}, \delta, y)
\end{bmatrix}
\]

\[
f(\hat{\psi}, \hat{\hat{r}}, \delta, y) = \hat{\hat{r}} - \left(4 - \frac{1}{T}\right)(\hat{\psi} - y)
\]

\[
g(\hat{\psi}, \hat{\hat{r}}, \delta, y) = -\frac{1}{T}\hat{\hat{r}} + \frac{K}{T}\delta - \left(2 - \frac{1}{T}\right)^2(\psi - y)
\]

To implement \( f \) and \( g \) we’ll build sub-circuits whose inputs are \( \hat{\psi}, \hat{\hat{r}}, \delta, \) and \( y \). We have all the basic components required to implement it:

We can build \( g \) similarly:
We now need to connect an integrator component to the output of $f$ and connect the integrator’s output to the $\hat{\psi}$ input of $f$ and $g$, (and similarly with the output of $g$). Now connect $y$ and $\delta$ to the corresponding inputs of $f$ and $g$.

4. Observability

Consider the following continuous time system.

$$
\dot{x}(t) = Ax(t) + Bu(t)
$$

$$
y(t) = Cx(t)
$$

(1)

We want to construct an estimate $z$ of the system state $x$. To do so, we construct a pretend system with the same $[A, B, C, D]$ models, the same input and the output of the last system along with an $L$ system matrix. We do this to try and exploit the difference between the output of our pretend state and the actual output, with $L$ being the “knob” that we can control.

$$
\dot{z}(t) = Az(t) + Bu(t) - L(Cz(t) - y(t))
$$

(2)

Define $e(t) = z(t) - x(t)$. This is the error term as a function of time.

(a) Using the two systems defined above, construct a system of the form,

$$
\frac{de}{dt}(t) = (A - LC)e(t)
$$

(3)

**Solution:** Subtracting (1) from (2), we get the desired answer.

(b) We want,

$$
\lim_{t \to \infty} e(t) = 0
$$

What does that imply about (3)?

**Solution:** This means that we want (3) to be stable.

(c) Does the initial value of the guess $z(0)$ matter in the long term?

**Solution:** Not really, since the $e(t)$ tends to 0. This means that, no matter how bad our initial guess, we will eventually have a good estimate.

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