EE16B - Spring'17 - Lecture 11B Notes[1]

Murat Arcak

6 April 2017

Interpolation with Basis Functions

Recall that interpolation aims to find a function that exactly matches data points (x_i, y_i), $i = 1, 2, \ldots$ When the data are evenly spaced, *i.e.*,

$$x_{i+1} - x_i = \Delta \quad \text{for all } i,$$

we can define a function $\phi(\cdot)$ such that

$$\phi(0) = 1 \quad \text{and} \quad \phi(k\Delta) = 0 \text{ when } k \neq 0 \tag{1}$$

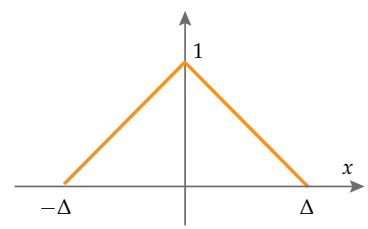
and interpolate between the data points with the function:

$$y = \sum_k y_k \phi(x - x_k). \tag{2}$$

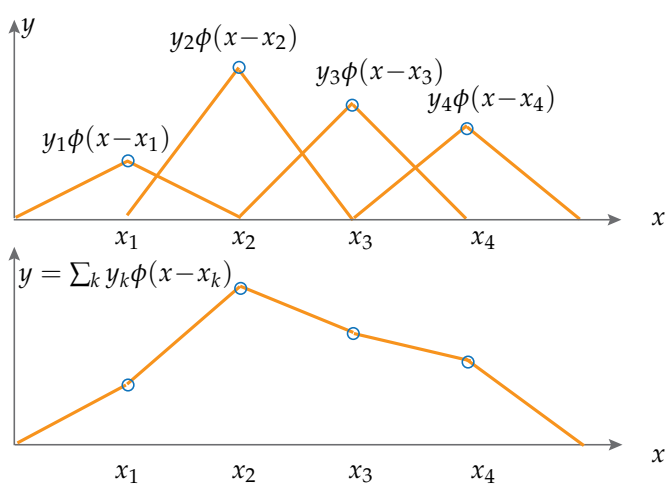
Linear Interpolation

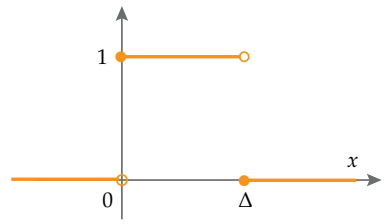
When $\phi(\cdot)$ is as depicted on the right, that is

$$\phi(x) = \begin{cases} 1 - \frac{|x|}{\Delta} & |x| \leq \Delta \\ 0 & \text{otherwise,} \end{cases}$$



then the interpolation (2) connects the data points with straight lines, as illustrated below.



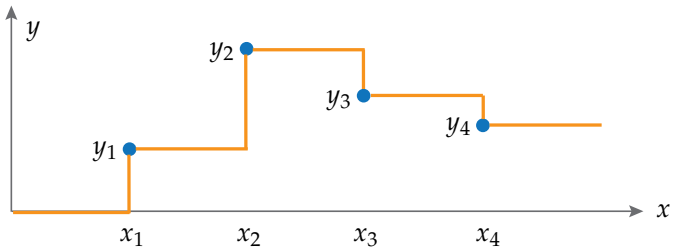


Zero Order Hold Interpolation

When

$$\phi(x) = \begin{cases} 1 & x \in [0, \Delta) \\ 0 & \text{otherwise} \end{cases}$$

as depicted on the right, the interpolation (2) keeps y constant between the data points:

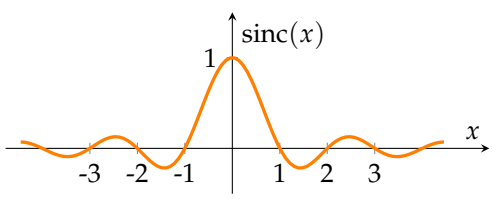


Sinc Interpolation

The *sinc* function is defined as

$$\text{sinc}(x) \triangleq \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

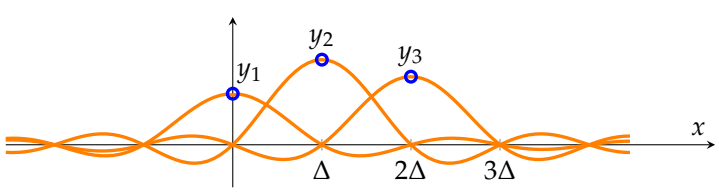
and depicted below. It is continuous since $\lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} = 1$, and vanishes whenever x is a nonzero integer.

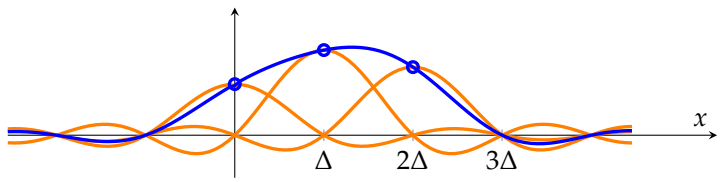


In sinc interpolation we apply (2) with

$$\phi(x) = \text{sinc}(x/\Delta).$$

To illustrate this, the first plot below superimposes $y_i \phi(x - x_i)$ for three data points $i = 1, 2, 3$. The second plot adds them up (blue curve) to complete the interpolation.





The interest in sinc interpolation is due to its smoothness – contrast the blue curve above with the kinks of linear interpolation and the discontinuities of zero order hold interpolation.

To make this smoothness property more explicit we use the identity

$$\text{sinc}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(\omega x) d\omega \tag{3}$$

which you can verify by evaluating the integral. Viewing this integral as an infinite sum of cosine functions, we see that the fastest varying component has frequency $\omega = \pi$. Thus the sinc function can't exhibit variations faster than this component.

Functions that involve frequencies smaller than some constant are called "band-limited." This notion is made precise in EE 120 with continuous *Fourier Transforms*. For 16B it is sufficient to think of a band-limited signal as one that can be written as a sum or integral of sinusoidal components whose frequencies lie in a finite band, which is $[0, \pi]$ for the sinc function in (3).

Sampling Theorem

Sampling is the opposite of interpolation: given a function $f(\cdot)$ we evaluate it at sample points x_i:

$$y_i = f(x_i) \quad i = 1, 2, 3, \ldots$$

Sampling is critical in digital signal processing where one uses samples of an analog sound or image. For example, digital audio is often recorded at 44.1 kHz which means that the analog audio is sampled 44,100 times per second; these samples are then used to reconstruct the audio when playing it back. Similarly, in digital images each pixel corresponds to a sample of an analog image.

An important problem in sampling is whether we can perfectly recover an analog signal from its samples. As we explain below, the answer is yes when the analog signal is band-limited and the interval between the samples is sufficiently short.

Suppose we sample the function $f : \mathbb{R} \to \mathbb{R}$ at evenly spaced points

$$x_i = \Delta i, \quad i = 1, 2, 3, \ldots$$

and obtain

$$y_i = f(\Delta i) \quad i = 1, 2, 3, \ldots$$

Then sinc interpolation between these data points gives:

$$\hat{f}(x) = \sum_i y_i \phi(x - \Delta i) \tag{4}$$

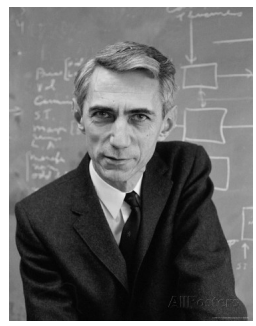
where

$$\phi(x) = \text{sinc}(x/\Delta),$$

which is band-limited by π/Δ from (3). This means that $\hat{f}(x)$ in (4) contains frequencies ranging from 0 to π/Δ.

Now if $f(x)$ involves frequencies smaller than π/Δ, then it is reasonable to expect that it can be recovered from (4) which varies as fast as $f(x)$. In fact the shifted sinc functions $\phi(x - \Delta i)$ form a basis for the space of functions[2] that are band-limited by π/Δ and the formula (4) is simply the representation of $f(x)$ with respect to this basis.

[2] for technical reasons this space is also restricted to square integrable functions



Claude Shannon (1916-2001)

Harry Nyquist (1889-1976)

Sampling Theorem: If $f(x)$ is band-limited by frequency

$$\omega_{\max} < \frac{\pi}{\Delta} \tag{5}$$

then the sinc interpolation (4) recovers $f(x)$, that is $\hat{f}(x) = f(x)$.

By defining the sampling frequency $\omega_s = 2\pi/\Delta$, we can restate the condition (5) as:

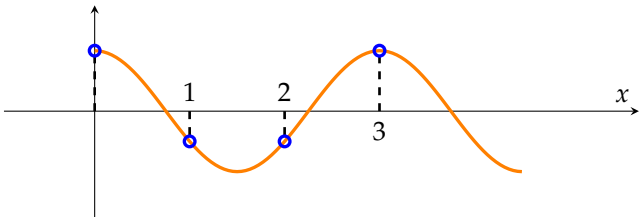
$$\omega_s > 2\omega_{\max}$$

which states that the function must be sampled faster than twice its maximum frequency. The Sampling Theorem was proven by Claude Shannon in the 1940s and was implicit in an earlier result by Harry Nyquist. Both were researchers at the Bell Labs.

Example 1: Suppose we sample the function

$$f(x) = \cos\left(\frac{2\pi}{3}x\right)$$

with period $\Delta = 1$. This means that we take 3 samples in each period of the cosine function, as shown in the figure below. Since $\omega_{\max} = \frac{2\pi}{3}$ and $\Delta = 1$, the criterion (5) holds and we conclude that the sinc interpolation (4) exactly recovers $f(x)$.



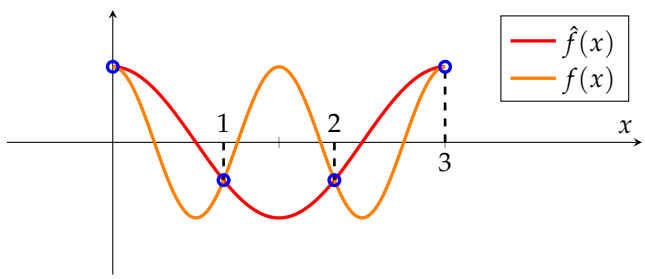
Example 2: Suppose now the function being sampled is

$$f(x) = \cos\left(\frac{4\pi}{3}x\right). \tag{6}$$

With $\omega_{\max} = \frac{4\pi}{3}$ and $\Delta = 1$, the criterion (5) fails. To see that the result of the sinc interpolation $\hat{f}(x)$ is now different from $f(x)$, note that this time we take 3 samples every *two* periods of the cosine function, as shown below. These samples are identical to the 3 samples collected in one period of the function in Example 1 above. Therefore, sinc interpolation gives the same result it did in Example 1:

$$\hat{f}(x) = \cos\left(\frac{2\pi}{3}x\right)$$

which does not match (6).



Aliasing and Phase Reversal

In Example 2 the low frequency component $2\pi/3$ appeared in $\hat{f}(x)$ from the actual frequency $4\pi/3$ of $f(x)$ that exceeded the critical value $\pi/\Delta = \pi$. The emergence of phantom lower frequency components as a result of under-sampling is known as "aliasing."

To generalize Example 2 suppose we sample the function

$$f(x) = \cos(\omega x + \phi) \tag{7}$$

with period $\Delta = 1$ and obtain

$$y_i = \cos(\omega i + \phi).$$

Using the identity $\cos(2\pi i - \theta) = \cos(\theta)$ which holds for any integer i, and substituting $\theta = \omega i + \phi$. we get

$$y_i = \cos(2\pi i - \omega i - \phi) = \cos((2\pi - \omega)i - \phi)$$

which suggests that the samples of the function

$$\cos((2\pi - \omega)x - \phi) \tag{8}$$

are identical to those of (7).

If $\omega \in (\pi, 2\pi]$ in (7) then sinc interpolation gives the function in (8) whose frequency is $2\pi - \omega \in [0, \pi)$. This function changes more slowly than (7) and the sign of the phase ϕ is reversed. These effects are visible in movies where a rotating wheel appears to rotate more slowly and in the opposite direction when its speed exceeds half of the sampling rate (18-24 frames/second).

Example 3: Suppose we sample the function

$$f(x) = \sin(1.9\pi x)$$

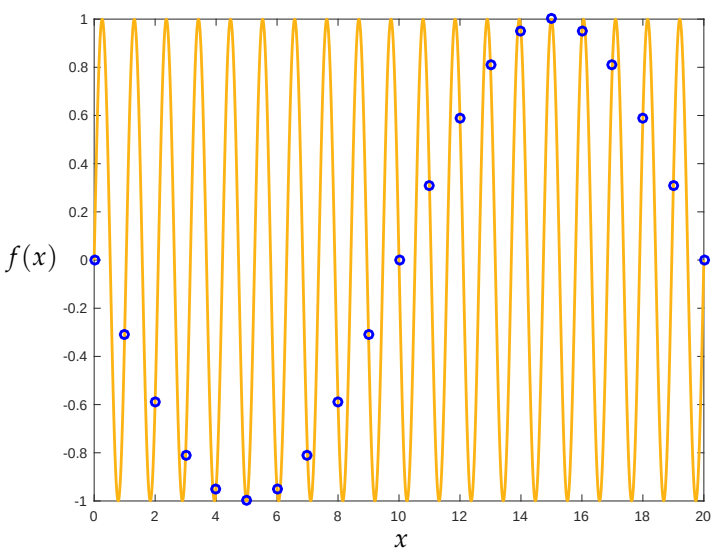
with $\Delta = 1$ as shown in the figure below. This function is of the form (7) with $\omega = 1.9\pi$ and $\phi = -\pi/2$ because

$$\sin(1.9\pi x) = \cos(1.9\pi x - \pi/2).$$

Thus, from (8), the sinc interpolation gives

$$\hat{f}(x) = \cos(0.1\pi x + \pi/2) = -\sin(0.1\pi x)$$

as evident from the samples in the figure. Note that the negative sign of $-\sin(0.1\pi x)$ is a result of the phase reversal discussed above.



Example 4: Note that the inequality in (5) is strict. To see that the Sampling Theorem does not hold when $\omega_{\max} = \frac{\pi}{\Delta}$ suppose $f(x) = \sin(\pi x)$ and $\Delta = 1$. Then the samples are

$$y_i = \sin(\pi i) = 0, \quad i = 1, 2, 3, \ldots$$

Since all samples are zero, sinc interpolation gives

$$\hat{f}(x) = 0$$

and does not recover $f(x) = \sin(\pi x)$.