Basis Signals and the Discrete Fourier Transform (DFT)

In the previous lecture we interpreted a finite-duration discrete-time signal \( x(t), t = 0, 1, \ldots, N - 1 \), as a vector

\[
\vec{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}
\]

and used this interpretation to define the inner product of two signals \( x(t) \) and \( y(t) \) as

\[
\vec{x}^T \vec{y} = \sum_{t=0}^{N-1} x(t)y(t).
\]

We then discussed orthogonal basis functions, such as the DCT basis\(^2\)

\[
v_0(t) = \sqrt{\frac{1}{N}}, \quad v_k(t) = \sqrt{\frac{2}{N}} \cos \left( \frac{\pi k}{2N} (2t + 1) \right) \quad k = 1, \ldots, N - 1,
\]

the first four of which \((k = 0, 1, 2, 3)\) are depicted below for \( N = 16 \).
Given orthonormal basis vectors $\vec{v}_0, \ldots, \vec{v}_{N-1}$ we can express any vector $\vec{x}$ as a linear combination
\[
\vec{x} = a_0\vec{v}_0 + \cdots + a_{N-1}\vec{v}_{N-1}
\]  
and obtain the coefficients from the inner product
\[
a_k = \vec{v}_k^T \vec{x}.
\]
This follows from orthonormality: if we multiply both sides of (1) by $\vec{v}_k^T$ all terms in the right hand side vanish, except $a_k \vec{v}_k^T \vec{v}_k = a_k$.

Similarly, given $N$ orthonormal basis signals $v_0(t), \ldots, v_{N-1}(t)$ we can express any signal $x(t)$ as a linear combination
\[
x(t) = a_0 v_0(t) + \cdots + a_{N-1} v_{N-1}(t) \quad t = 0, \ldots, N - 1
\]
and obtain the coefficients $a_k$ from the inner product
\[
a_k = \sum_{t=0}^{N-1} v_k(t) x(t).
\]
We can then store the coefficients $a_0, \ldots, a_{N-1}$ instead of the signal values $x(0), \ldots, x(N - 1)$. This is the premise of compression algorithms that store only the coefficients corresponding to basis functions that are important for the quality of sound, image, etc.

The Discrete Fourier Transform (DFT) is similar in spirit to the Discrete Cosine Transform but, as we will see, it uses complex valued basis signals. Therefore, before proceeding to DFT, we adapt the definition of inner products to complex vectors.

**Complex Inner Products**

For complex valued vectors $\vec{x}$ and $\vec{y}$ the appropriate inner product is
\[
\vec{x}^* \vec{y}
\]
where $\vec{x}^*$ is the conjugate transpose which means that, in addition to transposing, we take the complex conjugate. As an illustration,
\[
\vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix} \quad \Rightarrow \quad \vec{x}^T = \begin{bmatrix} 1 & j \end{bmatrix} \quad \vec{x}^* = \begin{bmatrix} 1 & -j \end{bmatrix}.
\]  
For a real-valued $\vec{x}$ there is no difference between $\vec{x}^*$ and $\vec{x}^T$, as the complex conjugate of a real number is itself. Note that
\[
\vec{x}^* \vec{x} = ||\vec{x}||^2
\]
which follows because $\vec{x}^* \vec{x} = \sum_i x(i)^* x(i) = \sum_i |x(i)|^2$. For the example in (2), $\vec{x}^* \vec{x} = 1 - j^2 = 2$ which means $||\vec{x}|| = \sqrt{2}$. By contrast, $\vec{x}^T \vec{x} = 1 + j^2 = 0$ which shows the necessity of conjugation when defining complex inner products.
Now if $\tilde{v}_0, \ldots, \tilde{v}_{N-1}$ are complex valued and orthonormal, that is
\[
\tilde{v}_k^* \tilde{v}_m = \begin{cases} 
1 & \text{if } k = m \\
0 & \text{if } k \neq m,
\end{cases}
\]
then the coefficients in the decomposition (1) become
\[
\alpha_k = \tilde{v}_k^* \tilde{x}.
\] (3)

**Discrete Fourier Transform (DFT)**

The DFT uses the basis signals
\[
u_k(t) \triangleq \frac{1}{\sqrt{N}} e^{j k \omega t}, \quad k = 0, 1, \ldots, N - 1 \quad \text{where} \quad \omega = \frac{2\pi}{N}
\] (4)
which we can rewrite as\(^3\)
\[
u_k(t) = \frac{1}{\sqrt{N}} \cos(k \omega t) + j \frac{1}{\sqrt{N}} \sin(k \omega t).
\]

The DFT basis is similar to DCT in that it consists of sinusoids of varying frequencies, but differs due to its complex values. The interest in DFT is because of computational efficiency\(^4\) and, as we will see in subsequent lectures, because it facilitates the analysis of linear time-invariant systems.

Finding the DFT of $x(t)$ means finding coefficients $X(k), k = 0, 1, \ldots, N - 1$, to represent $x(t)$ as a linear combination of $u_k(t)$:
\[
x(t) = \sum_{k=0}^{N-1} X(k) u_k(t).
\] (5)

**Example ($N = 2$):** Consider the signal
\[
x(0) = 2, \quad x(1) = 3.
\]
Since the duration is $N = 2$ we have $\omega = \pi$,
\[
u_0(t) = \frac{1}{\sqrt{2}} e^{j0t} = \frac{1}{\sqrt{2}} \quad \text{and} \quad u_1(t) = \frac{1}{\sqrt{2}} e^{j\pi t} = \frac{1}{\sqrt{2}} (-1)^t.
\]
We view $x(t)$, $u_0(t)$, $u_1(t)$ as length-two vectors whose entries are the values that each sequence takes at $t = 0, 1$:
\[
\tilde{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \tilde{u}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \tilde{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]
Then DFT becomes a change of basis
\[
\tilde{x} = X(0) \tilde{u}_0 + X(1) \tilde{u}_1
\]
similar to (1). Since \( \vec{u}_0 \) and \( \vec{u}_1 \) are real and orthonormal, that is
\[
\vec{u}_0^T \vec{u}_1 = 0 \quad \text{and} \quad \vec{u}_0^T \vec{u}_0 = \vec{u}_1^T \vec{u}_1 = 1,
\]
we get
\[
X(0) = \vec{u}_0^T \vec{x} = \frac{5}{\sqrt{2}}, \quad X(1) = \vec{u}_1^T \vec{x} = -\frac{1}{\sqrt{2}}.
\]

Example \((N = 4)\): When the duration is \( N = 4 \) we have \( \omega = \frac{2\pi}{N} = \frac{\pi}{2} \):
\[
\begin{align*}
\left(u_0(t) = \frac{1}{2} e^{j0t} = \frac{1}{2} \right) \\
\left(u_1(t) = \frac{1}{2} e^{j\frac{\pi}{2}t} = \frac{1}{2} (j)^t \right) \\
\left(u_2(t) = \frac{1}{2} e^{j\frac{\pi}{2}t} = \frac{1}{2} (-1)^t \right) \\
\left(u_3(t) = \frac{1}{2} e^{j\frac{3\pi}{2}t} = \frac{1}{2} (j)^t \right).
\end{align*}
\]
We view \( u_0(t), \ldots, u_3(t) \) as length-four vectors whose entries are the values that each sequence takes at \( t = 0, 1, 2, 3 \):
\[
\begin{align*}
\vec{u}_0 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ j \\ -1 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{u}_3 = \frac{1}{2} \begin{bmatrix} 1 \\ -j \\ -1 \end{bmatrix}.
\end{align*}
\]
To find \( X(k) \), \( k = 0, 1, 2, 3 \), such that
\[
\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = X(0) \vec{u}_0 + X(1) \vec{u}_1 + X(2) \vec{u}_2 + X(3) \vec{u}_3
\]
we will again use the orthonormality of the basis vectors \( \vec{u}_0, \ldots, \vec{u}_3 \).

You can indeed show that
\[
\vec{u}_k^* \vec{u}_m = \begin{cases} 0 & k \neq m \\ 1 & k = m. \end{cases}
\]
Using orthonormality as in (3) we can find the coefficients \( X(k) \) from
\[
X(k) = \vec{u}_k^* \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}
\]
and combine this equation for \( k = 0, 1, 2, 3 \) into
\[
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}
\]
where the first row of is \( \vec{u}_0^* \), the second row is \( \vec{u}_1^* \), and so on.
As an illustration, for the sequence $x(0) = 1, x(1) = x(2) = x(3) = 0$, we find

$$X(0) = X(1) = X(2) = X(3) = \frac{1}{2}.$$ 

The summation of $u_0(t), \ldots, u_3(t)$ with these weights indeed recovers $x(t)$ as shown in the figure below.

---

**Orthonormality of the DFT Basis**

In both examples ($N = 2$ and $N = 4$) we made use of the orthonormality of the DFT basis. We now show that this is a general property that holds for any $N$. To simplify the notation we define

$$W_k \triangleq e^{jk \frac{2\pi}{N}}$$

and rewrite (4) as

$$u_k(t) \triangleq \frac{1}{\sqrt{N}} W_k^t, \quad k = 0, 1, \ldots, N - 1,$$
or as the vector

\[ \vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ W_k \\ W_k^2 \\ \vdots \\ W_k^{(N-1)} \end{bmatrix}. \]

To see that these vectors form an orthonormal basis, note that

\[ \vec{u}_k^* \vec{u}_m = \frac{1}{N} \sum_{t=0}^{N-1} (W_k^*)^t W_m^t = \frac{1}{N} \sum_{t=0}^{N-1} e^{i(m-k)\frac{2\pi}{N} t} = \frac{1}{N} \sum_{t=0}^{N-1} W_{m-k}^t \]

where the second equality follows from \((W_k^*)^t = (e^{jk\frac{2\pi}{N} t})^* = e^{-jk\frac{2\pi}{N} t} \).

Now, if \( m = k \), \( e^{i(m-k)\frac{2\pi}{N} t} = 1 \) and the summation gives \( N \). If \( m \neq k \) the summation gives zero because the numbers \( W_{m-k}^t, t = 0, \ldots, N-1 \) are spread evenly around the unit circle and add up to zero, as illustrated on the right for \( m - k = 1 \) and \( N = 6 \). Thus,

\[ \vec{u}_k^* \vec{u}_m = \begin{cases} 0 & k \neq m \\ 1 & k = m. \end{cases} \]