Now that we know how simple CMOS gates operate, we could ask many questions:

- how fast do they switch?
- how much energy do they consume when switching (even when not switching)?
- how much space do they take up?

We can tackle the first question, but first, a review.

IIa. Capacitors

Any time we have two conductors separated by a dielectric (i.e. an insulator), we have the potential to separate charge across the two conductors.

This: 1) is called a capacitance (the device is a capacitor) 2) phenomenon stores energy

\[ \mathcal{C} \cdot \mathbf{v}_C \]

Symbol and definitions

The following is always true:

1) \( i_c(t) = \frac{C}{dt} \) [amps]

2) \( v_c(t) \) can never change instantly (i.e. it
can never be discontinuous. To see why, examine 1). If \( V_c(t) \) is discontinuous, then \( \frac{dV_c}{dt} \rightarrow \infty \) at the discontinuity and \( i_c \rightarrow \infty \).

3) The energy stored in a capacitor at any instant in time is

\[
U = \frac{1}{2} CV_c^2(t) \quad \text{[Joules]}
\]

**IIb. Capacitances and Transistors**

It turns out that whenever we make a transistor, there are always capacitances associated with the nodes. This is unavoidable and largely unwanted and arises from solid state physics.

In this class, we will model those capacitances as a single capacitor between \( V_{out} \) and \( GND \).

This is a simple model, but it works surprisingly well.

We are not telling you how you calculate what \( C_{out} \) is if given transistors, that is for another semester. We just "lump" all capacitive effects into one \( C_{out} \).

**IIc. The RC Circuit**
• Let’s look at the inverter now carefully now.

• Let’s assume $V_{in} = 0$ and thus $V_{out} = V_{DD}$ and the transistor has been in this condition a long time (since $t = -\infty$).

• Now, at $t = 0$, we instantly switch $V_{in}$ to $V_{DD}$. We know, eventually, $V_{out} = 0$. If there were no $C$, the output would change instantly because the equations have no concept of time. But with a $C$, things slow down. Let’s see why.

• For $t > 0$

Notice $V_c(t) = V_{out}(t)$ and $i = -i_c = -C \frac{dv_c}{dt}$

Use KVL: $V_c = V_R$

$V_c - V_R = 0$

$V_c - iR = 0$

$V_c - \left(-C \frac{dv_c}{dt}\right)R = 0$
\[ V_c + RC \frac{dV_c}{dt} = 0 \]
equivalently:
\[ V_{out} + RC \frac{dV_{out}}{dt} = 0 \]

This is a **first order differential equation**.
The general form is
\[ \frac{dx}{dt} + ax = 0 \]

Because, in this form, the RHS. = 0, it is a **homogenous 1st order D.E.**

(I sometimes use a shorthand whereby
\[ \frac{dx}{dt} = \dot{x} \quad \text{and} \quad \frac{d^2x}{dt^2} = \ddot{x} \quad \text{and so on.} \]

\[ V_{out} + \frac{V_{out}}{RC} = 0 \]

**IMPORTANT!!!**

You will discuss how to solve this class of equations in **DISCUSSION** using **linear algebra concepts. Learn the method!!!**
linear algebra concepts. Learn the method!!!

The solution to \( \frac{dV_{\text{out}}}{dt} + \frac{V_{\text{out}}}{RC} = 0 \)

for \( t > 0 \) is

\[
V_{\text{out}}(t) = V_{\text{out}}(0) e^{-\frac{t}{RC}}
\]

This is also called \( \tau \) (tau). \( \tau = RC \) and determines how fast the exponential decays.

this is the voltage across \( C \) at \( t = 0 \). Remember, for \( t < 0 \), \( V_{\text{out}} = V_{\text{dd}} \) and the voltage across a capacitor can never change instantly. Thus,

\[
V_{\text{out}}(0) = V_{\text{dd}}
\]

For our transistor problem, then:

\[
V_{\text{out}}(t) = V_{\text{dd}} e^{-\frac{t}{\tau}} \quad \text{where} \quad \tau = RC
\]
II d. Natural Response of a Charged Capacitor

\[ i(t) = \frac{C}{R} \frac{dv}{dt} = \frac{C}{R} \left( V_S - e^{-t/\tau} \right) \]

\[ = -\frac{V_S}{\tau} e^{-t/\tau} \quad (t > 0) \]
IIe. What about a general solution for RC circuits w/ DC sources?

This is the case we saw:

\[ v_c(t) = V_S e^{-t/\tau} \]

for \( t \geq 0 \):

\[ V_c(t) = RC \frac{dv_c}{dt} = 0 \]

Consider this:

\[ v_c(0) = V_S e^{-t/\tau} \]

\[ \text{KVL for } t > 0 \]
\[ \text{KVL for } V > 0 \]
\[ -V_{S2} + iR + v_c = 0 \]
\[ i = \frac{Cdv_c}{dt} \]
\[ -V_{S2} + RC \frac{dv_c}{dt} + v_c = 0 \]

\[ v_c + RC \frac{dv_c}{dt} = V_{S2} \]

General form of the inhomogeneous ODE

\[ \frac{dv_c}{dt} + \frac{v_c}{RC} = \frac{V_{S2}}{RC} \]

\[ v_c + av_c = b \]

What is the solution to this general ODE?

Two ways to write the solution:

1. \[ v_c(t) = v_c(0)e^{-t/RC} + \frac{b}{a} \left( 1 - e^{-t/RC} \right) \]

This is the decaying exponential of the initial voltage "discharging" up to \( v_{S2} \)

This is the exponential of the capacitor charging
Notice that at \( t \to \infty \), the capacitor will look like an open (because there is only a DC source) so by KVL \( V_c(\infty) = V_{sz} \).

In other words, the solution is the linear superposition of the initial charge of the capacitor discharging and the \( V_{sz} \) supply charging up to \( V_{sz} \).

\[
V_{sz}(1-e^{-t/RC}) \text{ looks like} \\
\uparrow \quad \cdots \cdots \cdots \\
V_{sz}
\]

\[\boxed{\mathbf{\Pi} \quad V_c(t) = V_c(\infty) + \left[ V(0) - V(\infty) \right] e^{-t/RC}} \]

This is equivalent to \( \mathbf{\Omega} \) but is often faster to write if you know \( V(0), V_c(\infty) \) and \( RC \).

All three of these are sometimes discoverable by inspection.

Just to be complete with terminology \( V_c(\infty) \) is the steady state and \( \left[ V(0) - V(\infty) \right] e^{-t/RC} \) is the transient.
transient