We now know about \( R \) and \( C \). We know \( C \)'s introduce "time" into our circuits.

**Why?**
1) The \( i-v \) relationship of capacitors has the concept of time in it:
\[
\dot{i} = C \frac{\mathrm{d}v}{\mathrm{d}t}
\]

2) Intuitively, capacitors store energy and any stored energy cannot be moved around infinitely fast. To do so would require infinite power, since
\[
\text{Power} = \frac{\mathrm{d}(\text{Energy})}{\mathrm{d}t}
\]

\[\hat{\text{I}}\] **Inductors**

Inductors store energy by setting up a magnetic field. In the same way that a capacitor separates charge \( Q \) and this leads to an \( E \) field, anytime we flow current down a conductor, this creates a magnetic field \( B \). Likewise, the magnetic field can store energy.

The magnitude of the magnetic field created by a straight wire is pretty small, so, we usually
use other geometries if we are trying to create a useful inductance on purpose.

A **solenoid** is a good example:

\[ L = \frac{N^2PA}{\mu l} \]

*Note:*

1) The inductance "\( L \)" depends on geometry and a material property called **permeability** (\( \mu \)).

2) The units of inductance are **Henry's** (\( H \)).

**II. Important circuit concepts of inductors**

1) \[ V_L(t) = L \frac{di_L(t)}{dt} \]

2) \( i_L(t) \) cannot change instantly

3) Inductors look like **short circuits** at DC (makes sense since they are wires)

4) The energy stored in an inductor is:

\[ U = \frac{1}{2} i_L^2(t) \]
III. RL Circuits

In the same way that one R and one C in a circuit can lead to an ODE and a solution with an \( \exp\left(-\frac{t}{RC}\right) \) term in it, an RL circuit results in an ODE and \( \exp\left(\frac{R}{L} t\right) \) terms. I will sketch an example below.

1. Note \( i_L \) for \( t < 0 \) is \( I_s \) because an inductor at DC looks like a short.

2. For \( t \geq 0 \), the circuit looks like:

3. Let's solve for \( i_L(t) \) for \( t \geq 0 \)

KVL: \[ V_L = V_R \]
\[ V_L = i_R R \]
\[ V_L = (-i_L) R \]
\[ V_L + i_L R = 0 \]
\[ V_L = L \frac{di_L}{dt} + i_L R = 0 \]

\[ \frac{di_L}{dt} + \frac{R}{L} i_L = 0 \]

\[ \text{Sol'n:} \quad 1\text{st order D.E.} \]

\[ i_L(t) = i_L(0) e^{-\left(\frac{R}{L}\right)t} \quad (A) \]

Since the current through the inductor cannot change instantly, \[ i_L(0) = I_0 \], so

\[ i_L(t) = I_0 e^{-\left(\frac{R}{L}\right)t} \quad (A) \]

\[ IV. \quad RLC \quad \text{Circuits} \]

We could spend some time solving RLC circuits, same as RC circuits and they are useful (they come up a lot for motors and power generation machinery. Also, RL's are a good way to generate giant voltages to cause air breakdown and sparks.
Look up how spark plugs work).

Here, however, we will focus on what happens when we put $R$, $L$ and $C$ together.

Consider:

1. For $t < 0$, $L$ looks like a short and $C$ looks like an open so $i_R = 0$. Thus, by KVL,
   \[ V_C(t < 0) = V_S \]
   \[ V_L(t < 0) = 0 \text{ (short)} \]
   \[ V_R(t < 0) = 0 \text{ (Because } i_R = 0) \]

2. Let's solve for $V_C(t)$ for $t \geq 0$

KVL: \[ V_L + V_C + V_R = 0 \]

\[ \frac{L}{L} \frac{di}{dt} + V_C + iR = 0 \]

Note:
\[ i = i_L = i_C = i_R \]
so
\[ i = \cap L \ldots \]
\[
\frac{L}{dt} \frac{di}{dt} + v_c + iR = 0
\]

so
\[
i = C \frac{dv_c}{dt}
\]

and
\[
v_L = L \frac{di}{dt}
\]

and
\[
v_R = iR
\]

(clean up)
\[
LC \frac{d^2 v_c}{dt^2} + RC \frac{dv_c}{dt} + v_c = 0
\]

Ack! Now we have a differential equation with a second derivative!

This is a 2\textsuperscript{nd} order D.E.

Because the R.H.S. = 0, it is called homogenous.

A general form is
\[
\ddot{v} + a\dot{v} + bv = 0
\]

Here, \( a = \frac{R}{L} \) and \( b = \frac{1}{LC} \).
Last lecture, we taught you a method for solving 2nd order ODE's using a very elegant technique that allows you to rewrite a 2nd order ODE as a system of coupled 1st order ODE's, which we can solve.

For now, we will just observe that the solution of
\[ \ddot{V} + a \dot{V} + bV = 0 \]

can be written as
\[ V(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} \]

where
\[ \lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

where
\[ \alpha = \frac{R}{2L} \text{ and } \omega_0 = \sqrt{\frac{1}{LC}} \]

and \[ K_1, K_2 \] are constants.

Notice that there are three possibilities for what type of number \( \lambda_1 \) and \( \lambda_2 \) can be.
1) If $\alpha > \omega_0$, then

\[
\lambda_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\
\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}
\]

this term is positive and $< \alpha$, so $\lambda_1$ and $\lambda_2$ are negative real numbers.

So the solution is just the sum of two decaying exponentials, $v(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$

2) If $\alpha < \omega_0$, then

\[
\lambda_1 = -\alpha + \sqrt{\omega_0^2 - \alpha^2} \\
\lambda_2 = -\alpha - \sqrt{\omega_0^2 - \alpha^2}
\]

this term is the square root of a negative number if $\alpha < \omega_0$

Re-write:

\[
\lambda_1 = -\alpha + j\omega_d \\
\lambda_2 = -\alpha - j\omega_d \\
\omega_d = \sqrt{\omega_0^2 - \alpha^2}
\]

What does this look like?

\[
v(t) = k_1 e^{(-\alpha + j\omega_d)t} + k_2 e^{(-\alpha - j\omega_d)t} \\
= k_1 e^{-\alpha t} e^{j\omega_d t} + k_2 e^{-\alpha t} e^{-j\omega_d t} \\
= e^{-\alpha t} (k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t})
\]
By Euler's formula, 
\[ e^{j\omega t} = \cos \omega t + j \sin \omega t \]
\[ e^{-j\omega t} = \cos \omega t - j \sin \omega t \]

So,
\[ V(t) = e^{-\alpha t} \left( k_1 \cos \omega t + j k_1 \sin \omega t + k_2 \cos \omega t - j k_2 \sin \omega t \right) \]
\[ = e^{-\alpha t} \left[ (k_1 + k_2) \cos \omega t + j(k_1 - k_2) \sin \omega t \right] \]

This looks like sinusoids with a frequency \( \omega \) slowly decaying as they are multiplied by \( e^{-\alpha t} \).

3) If \( \alpha = \omega_0 \), you get a special case. Then, \( \lambda_1 = \lambda_2 \Rightarrow \lambda \) and
\[ v(t) = k_1 e^{\alpha t} + k_2 e^{\alpha t} \]

1. Overdamped \( \alpha > \omega_0 \) Decays, no oscillation
2. Underdamped \( \alpha < \omega_0 \) Decays with oscillation
3. Critically damped \( \alpha = \omega_0 \) Decays, no oscillation (this is the boundary)

Notice that if \( R=0 \), i.e. there is no
clamping, then \[ \lambda = \pm j \omega \] and there is no decaying exponential.

That is, the circuit only oscillates at \( \omega_0 \) if \( R = 0 \) and then it does so forever.

In all other underdamped cases, the circuit oscillates at \( \omega_0 \) as it decays to 0.

**Example:** Consider the series RLC circuit above with the following values. Find \( v_c(t) \) and \( i(t) \).

\[
V_s = 16 \text{V} \quad R = 64 \Omega \quad L = 0.8 \text{H} \quad C = 2 \text{mF}
\]

\[
\alpha = \frac{R}{2L} = \frac{64}{2(0.8)} = 40 \text{ Nper} \quad \frac{\text{s}}{\text{s}}
\]

\[
\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.8 \times 2 \times 10^{-3}}} = 25 \text{ rad} \quad \frac{\text{s}}{\text{s}}
\]

\[ \therefore \lambda > \omega_0 \quad \text{and the circuit is overdamped.} \]

\[
v(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t}
\]

\[
\lambda_1 = -\alpha + j \sqrt{\omega_0^2 - \omega^2} = -40 + j\sqrt{40^2 - 25^2} = -8.8 \text{ Nper/s}
\]

\[
\lambda_2 = -\alpha - j \sqrt{\omega_0^2 - \omega^2} = -40 - j\sqrt{40^2 - 25^2} = -71.2 \text{ Nper/s}
\]

\[
v(t) = K_1 e^{-8.8t} + K_2 e^{-71.2t}
\]
Now we need to find $K_1$ and $K_2$. For this, we will use the initial conditions.

$\mathbf{v}(0) = \begin{bmatrix} K_1 + K_2 = 16 \end{bmatrix}$ (Remember that we know that $\mathbf{v}(t=0) = 16 \mathbf{V}$)

We need one more equation. We know

$i(t=0) = 0 \mathbf{A}$

We also know that $i_c(t) = \frac{C_d \mathbf{v}_c}{dt}$

$i_c(t) = -8.8K_1 e^{-8.8t} + -71.2K_2 e^{-71.2t}$

$i_c(0) = -8.8K_1 + -71.2K_2 = 0$

Solving this system, $K_1 = 18.2564$

$K_2 = -2.2564$

$\therefore \mathbf{v}(t) = 18.2564 e^{-8.8t} - 2.25641 e^{-71.2t}$ $[\mathbf{V}]$

$i(t) = -8.8(18.2564) e^{-8.8t} + (2.25641)(71.2) e^{-71.2t}$ $[\mathbf{A}]$