Lecture 2A_Sp2017

1/31/2017

We now know about R and C. We know C's introduce "time" into our circuits.

Why? 1) The i-v relationship of capacitors has the concept of time in it:

$$i = C\frac{dv}{dt}$$

2) Intuitively, capacitors store energy and any stored energy cannot be moved around infinitely fast. To do so would require infinite power, since Power $= \frac{d}{dt}$(Energy)

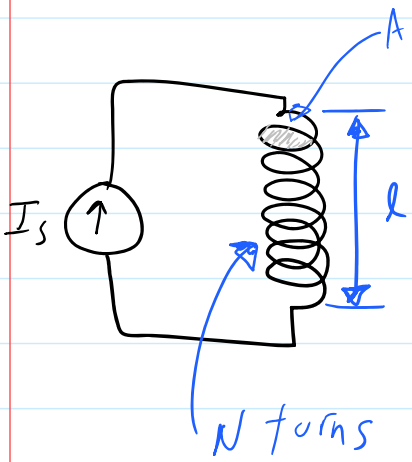
I. Inductors

Inductors store energy by setting up a magnetic field. In the same way that a capacitor separates charge(Q) and this leads to an $\vec{E}$ field, anytime we flow current down a conductor, this creates a magnetic field ($\vec{B}$). Likewise, the magnetic field can store energy.

The ~~magnitude~~ of the magnetic field created by a straight wire is pretty small, so we usually

use other geometries if we are trying to create a useful inductance on purpose.

A <u>solenoid</u> is a good example:



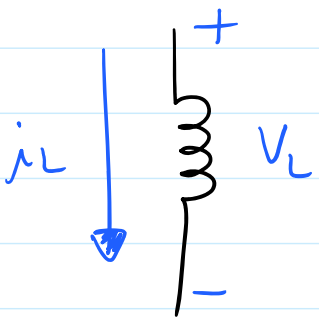
$$L = \frac{N^2 \mu A}{\ell} \quad [H]$$

<u>Note:</u>

1) The inductance "L" depends on <u>geometry</u> and a material property called <u>permeability</u> (μ).

2) The units of inductance are Henry's (H)

II. Important circuit concepts of inductors



1) $V_L(t) = L \frac{d i_L(t)}{dt}$

2) $i_L(t)$ cannot change instantly

3) inductors look like <u>short circuits</u> at DC (makes sense since they are wires) ☺

Notice all of this is the <u>dual</u> of the capacitor properties!!

4) The energy stored in an inductor is:

$U = \frac{1}{2} L i^2(t)$

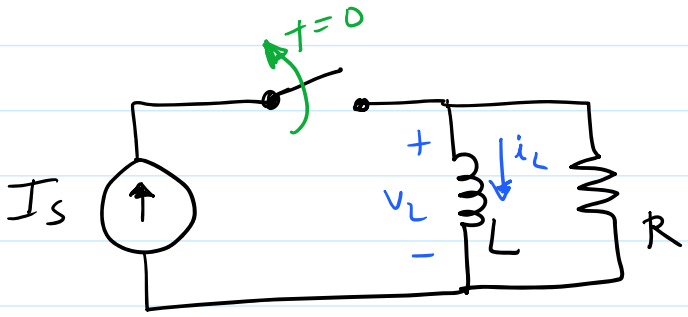
the prof

$$U_L = \frac{1}{2} L i_L^2(t)$$

III. RL Circuits

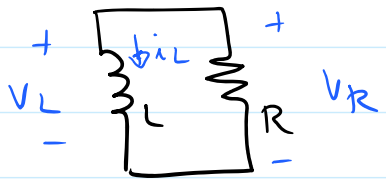
In the same way that one R and one C in a circuit can lead to an ODE and a solution with an $\exp\left(\frac{-t}{RC}\right)$ term in it,

an RL circuit results in an ODE and $\exp\left(\frac{-R}{L}t\right)$ terms. I will sketch an example below.



① Note i_L for $t<0$ is I_S because an inductor at DC looks like a short.

② For $t \geq 0$, the circuit looks like:



③ Let's solve for $i_L(t)$ for $t \geq 0$

KVL: $\quad v_L = v_R$

$$V_L = i_R R$$

$$V_L = (-i_L) R$$

$$V_L + i_L R = 0$$

$V_L = L \frac{di_L}{dt}$ →

$$L \frac{di_L}{dt} + i_L R = 0$$

look familiar?

$\dot{x} + ax = 0$ where $a = \frac{R}{L}$

$$\frac{di_L}{dt} + \frac{R}{L} i_L = 0$$

Soln: 1st order D.E.

$$i_L(t) = i_L(0)\, e^{-\left(\frac{R}{L}\right)t} \qquad (A)$$

Since the current through the inductor cannot change instantly, $i_L(0) = I_S$, so

$$i_L(t) = I_S\, e^{-\left(\frac{R}{L}\right)t} \qquad (A)$$

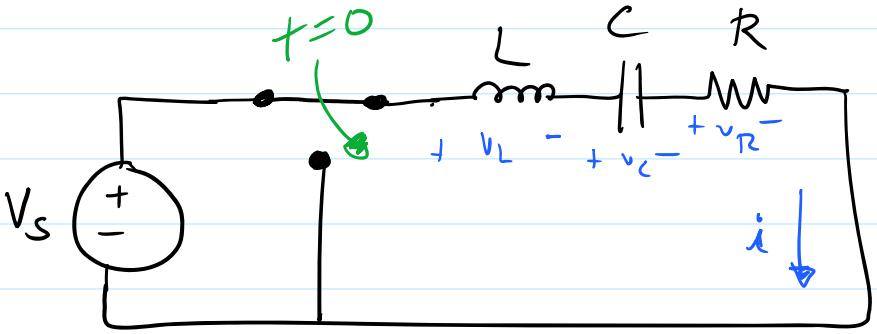
IV. RLC Circuits

We could spend some time solving RL circuits same as RC circuits and they are useful (they come up a lot for motors and power generation machinery. Also, RL's are a good way to generate giant voltages to cause air breakdown and sparks.

Look up how spark plugs work).

Here, however, we will focus on what happens when we put R, L and C together.

Consider:



$i = i_R = i_L = i_C$

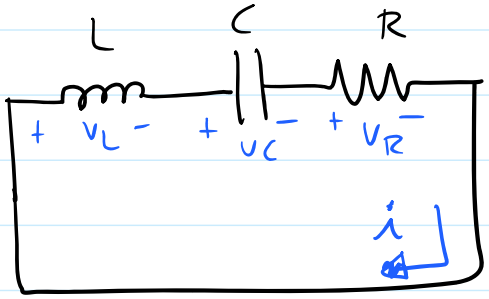
① For $t<0$, L looks like a short and C looks like an open so $i_R = 0$. Thus, by KVL,

$v_C(t<0) = V_S$

$v_L(t<0) = 0$ (short)

$v_R(t<0) = 0$ (Because $i_R = 0$)

② Let's solve for $v_C(t)$ for $t \geq 0$



KVL: $v_L + v_C + v_R = 0$

$$\frac{L\,di}{dt} + v_C + iR = 0$$

Note:

$i = i_L = i_C = i_R$

so

$i = C\frac{dv_C}{dt}$

$$\frac{L\,di}{dt} + v_c + iR = 0$$

$$L\frac{d}{dt}\left(C\frac{dv_c}{dt}\right) + v_c + \left(C\frac{dv_c}{dt}\right)R = 0$$

$$LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = 0$$

so

$$i = C\frac{dv_c}{dt}$$

and

$$v_L = L\frac{di}{dt}$$

and

$$v_R = iR$$

(clean up)

$$\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = 0$$

Ack! Now we have a differential equation with a second derivative!

This is a 2nd order D.E.

Because the R.H.S. $= 0$, it is called homogenous.

A general form is

$$\ddot{v} + a\dot{v} + bv = 0$$

Here, $a = \frac{R}{L}$ and $b = \frac{1}{LC}$

V. 2nd order O.D.E. solution

Last lecture, we taught you a method for solving 2nd order O.D.E's using a very elegant technique that allows you to rewrite a 2nd order ODE as a system of coupled 1st order ODE's, which we can solve.

For now, we will just observe that the solution of

$$\ddot{v} + a\dot{v} + bv = 0$$

can be written as

$$v(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t}$$

where $\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$ ← two roots, hence λ_1 and λ_2

where $\alpha = \frac{R}{2L}$ and $\omega_0 = \sqrt{\frac{1}{LC}}$

α and ω_0 depend only on circuit elements R, L, C

and K_1, K_2 are constants.

Notice that there are three possibilities for what type of number λ_1 and λ_2 can be.

1) If $\alpha > \omega_0$, then $\lambda_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

$\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

this term is positive and $< \alpha$, so

λ_1 and λ_2 are negative real numbers

So the solution is just the sum of two decaying exponentials, $v(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$

2) If $\alpha < \omega_0$, then $\lambda_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

$\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

this term is the square root of a negative number if $\alpha < \omega_0$

Re-write:

$$\lambda_1 = -\alpha + j\omega_D \qquad \omega_D = \sqrt{\omega_0^2 - \alpha^2}$$
$$\lambda_2 = -\alpha - j\omega_D$$

What does this look like?

$$v(t) = k_1 e^{(-\alpha + j\omega_D)t} + k_2 e^{(-\alpha - j\omega_D)t}$$

$$= k_1 e^{-\alpha t} e^{j\omega_D t} + k_2 e^{-\alpha t} e^{-j\omega_D t}$$

$$= e^{-\alpha t}\left(k_1 e^{j\omega_D t} + k_2 e^{-j\omega_D t}\right)$$

By Euler's formula, $e^{j\omega_D t} = \cos\omega_D t + j\sin\omega_D t$

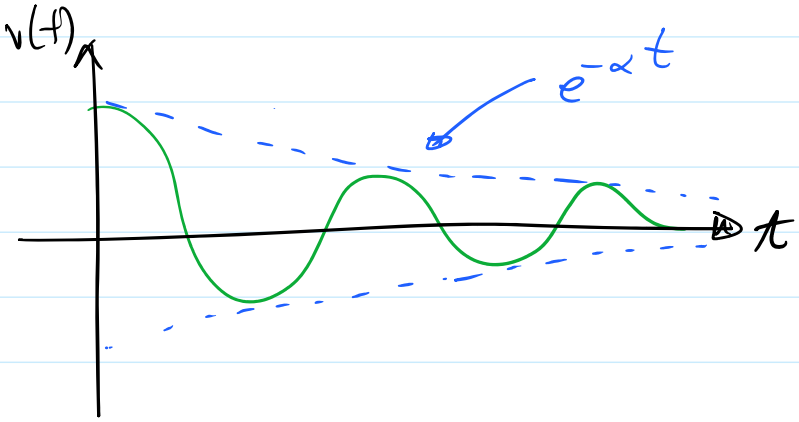
$e^{-j\omega_D t} = \cos\omega_D t - j\sin\omega_D t$

So,

$$v(t) = e^{-\alpha t}\left(K_1\cos\omega_D t + jK_1\sin\omega_D t + K_2\cos\omega_D t - jK_2\sin\omega_D t\right)$$

$$= e^{-\alpha t}\left[(K_1+K_2)\cos\omega_D t + j(K_1-K_2)\sin\omega_D t\right]$$

This looks like sinusoids with a frequency ω_D slowly decaying as they are multiplied by $e^{-\alpha t}$.



3) If $\alpha = \omega_0$, you get a special case. Then, $\lambda_1 = \lambda_2 \Rightarrow \lambda$ and $v(t) = K_1 e^{\lambda t} + K_2 t e^{\lambda t}$

1	Overdamped	$\alpha > \omega_0$	Decays, no oscillation
2	Underdamped	$\alpha < \omega_0$	Decays with oscillation
3	Critically damped	$\alpha = \omega_0$	Decays, no oscillation (this is the boundary)

Notice that if $R = 0$, i.e. there is no

damping, then $\lambda = \pm j\omega_0$ and there is no decaying exponential.

That is, the circuit only oscillates at ω_0 if $R=0$ and then it does so forever.

In all other underdamped cases, the circuit oscillates at ω_D as it decays to 0.

Example: Consider the series RLC above with the following values. Find $v_C(t)$ and $i(t)$.

$V_S = 16V \quad R = 64\Omega \quad L = 0.8H \quad C = 2mF$

$$\alpha = \frac{R}{2L} = \frac{64}{2(0.8)} = 40 \frac{Nepers}{s}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.8 \times 2 \times 10^{-3}}} = 25 \frac{rad}{s}$$

$\therefore \quad \alpha > \omega_0$ and the circuit is overdamped.

$$v(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t}$$

$$\lambda_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -40 + \sqrt{40^2 - 25^2} = -8.8 \ Np/s$$
$$\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -40 - \sqrt{40^2 - 25^2} = -71.2 \ Np/s$$

$$v(t) = K_1 e^{-8.8t} + K_2 e^{-71.2t}$$

Now we need to find K_1 and K_2. For this, we will use the initial conditions.

$v(0) = \boxed{K_1 + K_2 = 16}$ (Remember that we know that $v_c(t=0) = 16\,V$)

We need one more equation. We know $i(t=0) = 0\,A$

We also know that $i_c(t) = C\frac{dv_c}{dt}$

$$i_c(t) = -8.8K_1 e^{-8.8t} + -71.2K_2 e^{-71.2t}$$

$$i_c(0) = \boxed{-8.8K_1 + -71.2K_2 = 0}$$

Solving this system, $K_1 = 18.2564$

$K_2 = -2.25641$

$$\therefore \quad v(t) = 18.2564\, e^{-8.8t} - 2.25641\, e^{-71.2t} \quad [V]$$

$$i(t) = -8.8(18.2564)\, e^{-8.8t} + (2.25641)(71.2)\, e^{-71.2t} \quad [A]$$