Linearization

Linear models are advantageous because their solutions, stability properties, and stabilizing controllers can be studied using linear algebra. The methods applicable to nonlinear models are limited; therefore it is common practice to approximate a nonlinear model with a linear one that is valid around a desired operating point.

Recall that the Taylor approximation of a differentiable function $f$ around a point $x^*$ is:

$$f(x) \approx f(x^*) + \nabla f(x)\big|_{x=x^*} (x - x^*),$$

as illustrated on the right for a scalar-valued function of a single variable. When $x$ and $f(x)$ are $n$-vectors as in our state models, $\nabla f(x)$ must be interpreted as the $n \times n$ matrix of partial derivatives:

$$\nabla f(x) = \begin{bmatrix}
    \frac{\partial f_1(x_1, \ldots, x_n)}{\partial x_1} & \frac{\partial f_1(x_1, \ldots, x_n)}{\partial x_2} & \cdots & \frac{\partial f_1(x_1, \ldots, x_n)}{\partial x_n} \\
    \frac{\partial f_2(x_1, \ldots, x_n)}{\partial x_1} & \frac{\partial f_2(x_1, \ldots, x_n)}{\partial x_2} & \cdots & \frac{\partial f_2(x_1, \ldots, x_n)}{\partial x_n} \\
    \vdots & \vdots & \ddots & \vdots \\
    \frac{\partial f_n(x_1, \ldots, x_n)}{\partial x_1} & \frac{\partial f_n(x_1, \ldots, x_n)}{\partial x_2} & \cdots & \frac{\partial f_n(x_1, \ldots, x_n)}{\partial x_n}
\end{bmatrix}. $$

We linearize nonlinear state models by applying this approximation around an equilibrium point. For the continuous-time system

$$\frac{d}{dt} \bar{x}(t) = f(\bar{x}(t)), \quad (1)$$

$\bar{x}$ is called an equilibrium when $f(\bar{x}) = 0$ because, if the initial condition is $\bar{x}$, then $\frac{d}{dt} \bar{x}(t) = 0$ and $\bar{x}(t)$ remains at $\bar{x}$. If we define the deviation of $\bar{x}$ from $\bar{x}$ as:

$$\tilde{x}(t) \triangleq \bar{x}(t) - \bar{x}, \quad (2)$$

then we see that

$$\frac{d}{dt} \tilde{x}(t) = f(\bar{x}(t)) \approx f(\bar{x}) + \nabla f(\bar{x})\big|_{\bar{x}=\bar{x}} \tilde{x}(t).$$

Substituting $f(\bar{x}) = 0$ and defining

$$A \triangleq \nabla f(\bar{x})\big|_{\bar{x}=\bar{x}}, \quad (3)$$

\[f(x) \approx f(x^*) + \nabla f(x)\big|_{x=x^*} (x - x^*)\]
we obtain the linearization of (1) around the equilibrium $\vec{x}^*$:

$$\frac{d}{dt} \tilde{x}(t) \approx A\tilde{x}(t).$$

In the discrete-time case

$$\vec{x}(t+1) = f(\vec{x}(t)), $$

$\vec{x}^*$ is an equilibrium point if $f(\vec{x}^*) = \vec{x}^*$. The vector $\tilde{x}(t)$ defined in (2) satisfies:

$$\tilde{x}(t+1) = \tilde{x}(t)+ \vec{x}^* = f(\vec{x}(t)) - \vec{x}^* \approx f(\vec{x}^*) - \vec{x}^* + \nabla f(\vec{x})|_{\vec{x}=\vec{x}^*} \tilde{x}(t). $$

Substituting $f(\vec{x}^*) - \vec{x}^* = 0$ and defining $A$ as in (3), we get

$$\tilde{x}(t+1) \approx A\tilde{x}(t).$$

**Example:** Recall the pendulum model derived in Lecture 4A:

$$\frac{dx_1(t)}{dt} = x_2(t), $$

$$\frac{dx_2(t)}{dt} = -\frac{k}{m} x_2(t) - \frac{g}{l} \sin x_1(t) $$

where $x_1(t) \triangleq \theta(t)$ and $x_2(t) \triangleq \frac{d\theta(t)}{dt}$.

To find the equilibrium points note that

$$f(\vec{x}) = \begin{bmatrix} x_2 \\ -\frac{k}{m} x_2 - \frac{g}{l} \sin x_1 \end{bmatrix} = 0$$

when $x_2 = 0$ and $\sin x_1 = 0$. Thus the two distinct equilibrium points are the downward position:

$$x_1 = 0, \quad x_2 = 0, $$

and the upright position:

$$x_1 = \pi, \quad x_2 = 0. $$

Since the entries of $f(\vec{x})$ are $f_1(\vec{x}) = x_2$ and $f_2(\vec{x}) = -\frac{k}{m} x_2 - \frac{g}{l} \sin x_1$, we have

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} \cos x_1 & -\frac{k}{m} \end{bmatrix}. $$

By evaluating this matrix at (5) and (6), we obtain the linearization around the respective equilibrium point:

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}, \quad A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}. $$
Stability of Linear State Models

The Scalar Case

We first study a system with a single state variable \( x(t) \) that obeys

\[
x(t + 1) = ax(t) + bu(t)
\]  

(8)

where \( a \) and \( b \) are constants. If we start with the initial condition \( x(0) \), then we get by recursion

\[
x(1) = ax(0) + bu(0)
\]

\[
x(2) = ax(1) + bu(1) = a^2x(0) + abu(0) + bu(1)
\]

\[
x(3) = ax(2) + bu(2) = a^3x(0) + a^2bu(0) + abu(1) + bu(2)
\]

\[
\vdots
\]

\[
x(t) = a^t x(0) + a^{t-1}bu(0) + a^{t-2}bu(1) + \cdots + abu(t-2) + bu(t-1),
\]

rewritten compactly as:

\[
x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-1-k}bu(k) \quad t = 1, 2, 3, \ldots
\]

(9)

The first term \( a^t x(0) \) represents the effect of the initial condition and the second term \( \sum_{k=0}^{t-1} a^{t-1-k}bu(k) \) represents the effect of the input sequence \( u(0), u(1), \ldots, u(t-1) \).

**Definition.** We say that a system is **stable** if its state \( x(t) \) remains bounded for any initial condition and any bounded input sequence. Conversely, we say it is **unstable** if we can find an initial condition and a bounded input sequence such that \( |x(t)| \to \infty \) as \( t \to \infty \).

It follows from (9) that, if \( |a| > 1 \), then a nonzero initial condition \( x(0) \neq 0 \) is enough to drive \( |x(t)| \) unbounded. This is because \( |a|^t \) grows unbounded and, with \( u(t) = 0 \) for all \( t \), we get \( |x(t)| = |a^t x(0)| = |a|^t |x(0)| \to \infty \). Thus, (8) is unstable for \( |a| > 1 \).

Next, we show that (8) is stable when \( |a| < 1 \) is stable. In this case \( a^t x(0) \) decays to zero, so we need only to show that the second term in (9) remains bounded for any bounded input sequence. A bounded input means we can find a constant \( M \) such that \( |u(t)| \leq M \) for all \( t \).

Thus,

\[
\left| \sum_{k=0}^{t-1} a^{t-1-k}bu(k) \right| \leq \sum_{k=0}^{t-1} |a|^{t-1-k}|b||u(k)| \leq |b|M \sum_{k=0}^{t-1} |a|^{t-1-k}.
\]

Defining the new index \( s = t - 1 - k \) we rewrite the last expression as

\[
|b|M \sum_{s=0}^{t-1} |a|^s,
\]
and note that $\sum_{s=0}^{t-1} |a|^s$ is a geometric series that converges to $\frac{1}{1-|a|}$ since $|a| < 1$. Therefore, each term in (9) is bounded and we conclude stability for $|a| < 1$.

**Summary:** The scalar system (8) is stable when $|a| < 1$, and unstable when $|a| > 1$.

When $a$ is a complex number, a perusal of the stability and instability arguments above show that the same conclusions hold if we interpret $|a|$ as the modulus of $a$, that is:

$$|a| = \sqrt{\text{Re}\{a\}^2 + \text{Im}\{a\}^2}.$$

What happens when $|a| = 1$? If we disallow inputs ($b = 0$), this case is referred to as “marginal stability” because $|a'x(0)| = |x(0)|$, which neither grows nor decays. If we allow inputs ($b \neq 0$), however, we can find a bounded input to drive the second term in (9) unbounded. For example, when $a = 1$, the constant input $u(t) = 1$ yields:

$$\sum_{k=0}^{t-1} a^{t-1-k}bu(k) = \sum_{k=0}^{t-1} b = bt$$

which grows unbounded as $t \to \infty$. Therefore, $|a| = 1$ is a precarious case that must be avoided in designing systems.