EE 16B Midterm 2, March 21, 2017

Name: ________________
SID #: ________________
Discussion Section and TA: ________________
Lab Section and TA: ________________
Name of left neighbor: ________________
Name of right neighbor: ________________

Important Instructions:

- **Show your work.** An answer without explanation is not acceptable and does not guarantee any credit.
- **Only the front pages will be scanned and graded.** You can use the back pages as scratch paper.
- **Do not remove pages**, as this disrupts the scanning. Instead, cross the parts that you don’t want us to grade.

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1. (10 points) The thirteenth century Italian mathematician Fibonacci described the growth of a rabbit population by the recurrence relation:

\[ y(t + 2) = y(t + 1) + y(t) \]

where \( y(t) \) denotes the number of rabbits at month \( t \). A sequence generated by this relation from initial values \( y(0), y(1) \) is known as a Fibonacci sequence.

a) (5 points) Bring the recurrence relation above to the state space form using the variables \( x_1(t) = y(t) \) and \( x_2(t) = y(t + 1) \).
b) (5 points) Determine the stability of this system.
2. (15 points) Consider the circuit below that consists of a capacitor, an inductor, and a third element with the nonlinear voltage-current characteristic:

\[ i = -v + v^3. \]

![Circuit Diagram]

a) (5 points) Write a state space model of the form

\[
\frac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t)) \\
\frac{dx_2(t)}{dt} = f_2(x_1(t), x_2(t))
\]

using the states \( x_1(t) = v_C(t) \) and \( x_2(t) = i_L(t) \).

\[
f_1(x_1, x_2) = \boxed{} \quad f_2(x_1, x_2) = \boxed{}
\]
b) (5 points) Linearize the state model at the equilibrium $x_1 = x_2 = 0$ and specify the resulting $A$ matrix.
c) (5 points) Determine stability based on the linearization.
3. (10 points) Consider the discrete-time system

\[ \tilde{x}(t + 1) = A\tilde{x}(t) + Bu(t) \]

where

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad B = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}.
\]

a) (5 points) Determine if the system is controllable.
b) (5 points) Explain whether or not it is possible to move the state vector from $\vec{x}(0) = 0$ to

$$\vec{x}(T) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$ 

If your answer is yes, specify the smallest possible time $T$ and an input sequence $u(0), \ldots, u(T-1)$ to accomplish this task.
4. (20 points) Consider the system

\[ \vec{x}(t + 1) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \]

where \( \theta \) is a constant.

a) (5 points) For which values of \( \theta \) is the system controllable?

b) (10 points) Select the coefficients \( k_1, k_2 \) of the state feedback controller

\[ u(t) = k_1 x_1(t) + k_2 x_2(t) \]

such that the closed-loop eigenvalues are \( \lambda_1 = \lambda_2 = 0 \). Your answer should be symbolic and well-defined for the values of \( \theta \) you specified in part (a).
Additional workspace for Problem 4b.
c) (5 points) Suppose the state variable $x_1(t)$ evolves as depicted below when no control is applied ($u = 0$). What is the value of $\theta$?
5. (15 points) Consider the inverted pendulum below, where $p(t)$ is the position of the cart, $\theta(t)$ is the angle of the pendulum, and $u(t)$ is the input force.

When linearized about the upright position, the equations of motion are

\[
\begin{align*}
\ddot{p}(t) &= -\frac{m}{M}g\theta(t) + \frac{1}{M}u(t) \\
\ddot{\theta}(t) &= \frac{M+m}{M\ell}g\theta(t) - \frac{1}{M\ell}u(t)
\end{align*}
\]  

(1)

where $M$, $m$, $\ell$, $g$ are positive constants.

a) (5 points) Using (1) write the state model for the vector

\[\bar{x}(t) = \begin{bmatrix} p(t) & \dot{p}(t) & \theta(t) & \dot{\theta}(t) \end{bmatrix}^T.\]
b) (5 points) Suppose we measure only the position; that is, the output is \( y(t) = x_1(t) \). Determine if the system is observable with this output.
c) (5 points) Suppose we measure only the angle; that is, the output is \( y(t) = x_3(t) \). Determine if the system is observable with this output.
6. (15 points) Consider the system
\[
\begin{bmatrix}
    x_1(t+1) \\
    x_2(t+1) \\
    x_3(t+1)
\end{bmatrix} =
\begin{bmatrix}
    0.9 & 0 & 0 \\
    0 & 1 & 1 \\
    0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t)
\end{bmatrix}, \quad y(t) =
\begin{bmatrix}
    0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t)
\end{bmatrix}.
\]

Let \( A \) denote the matrix above.

a) (5 points) Select values for \( \ell_1, \ell_2, \ell_3 \) in the observer below such that \( \hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t) \) converge to the true state variables \( x_1(t), x_2(t), x_3(t) \) respectively.

\[
\begin{bmatrix}
    \hat{x}_1(t+1) \\
    \hat{x}_2(t+1) \\
    \hat{x}_3(t+1)
\end{bmatrix} =
\begin{bmatrix}
    0.9 & 0 & 0 \\
    0 & 1 & 1 \\
    0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
    \hat{x}_1(t) \\
    \hat{x}_2(t) \\
    \hat{x}_3(t)
\end{bmatrix} +
\begin{bmatrix}
    \ell_1 \\
    \ell_2 \\
    \ell_3
\end{bmatrix}
\begin{bmatrix}
    \ell_1 \\
    \ell_2 \\
    \ell_3
\end{bmatrix}
\begin{bmatrix}
    \hat{x}_2(t) - y(t)
\end{bmatrix}.
\]
Additional workspace for Problem 6a.
b) (5 points) Professor Arcak found a solution to part (a) that guarantees convergence of $\hat{x}_3(t)$ to $x_3(t)$ in one time step; that is

$$\hat{x}_3(t) = x_3(t) \quad t = 1, 2, 3, \ldots$$

for any initial $\vec{x}(0)$ and $\hat{x}(0)$. Determine his $\ell_3$ value based on this behavior of the observer. Explain your reasoning.
c) (5 points) When Professor Arcak solved part (a), he found the convergence of $\hat{x}_1(t)$ to $x_1(t)$ to be rather slow no matter what $L$ he chose. Explain the reason why no choice of $L$ can change the convergence rate of $\hat{x}_1(t)$ to $x_1(t)$. 
7. (15 points) Consider a system with the symmetric form

\[
\frac{d}{dt} \begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{bmatrix} = \begin{bmatrix} F & H \\ H & F \end{bmatrix} \begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{bmatrix} + \begin{bmatrix} G \\ G \end{bmatrix} \vec{u}(t),
\]

where \( \vec{x}_1 \) and \( \vec{x}_2 \) have identical dimensions and, therefore, \( F \) and \( H \) are square matrices.

a) (5 points) Define the new variables

\[
\vec{z}_1 = \vec{x}_1 + \vec{x}_2 \quad \text{and} \quad \vec{z}_2 = \vec{x}_1 - \vec{x}_2,
\]

and write a state model with respect to these variables:

\[
\frac{d}{dt} \begin{bmatrix} \vec{z}_1(t) \\ \vec{z}_2(t) \end{bmatrix} = \begin{bmatrix} \vec{z}_1(t) \\ \vec{z}_2(t) \end{bmatrix} + \begin{bmatrix} \vec{z}_1(t) \\ \vec{z}_2(t) \end{bmatrix} \vec{u}(t).
\]
b) (5 points) Show that the system (2) is not controllable.
c) (5 points) Write a state model for the circuit below using the inductor currents as the variables. Show that the model has the symmetric form (2).