1 Second-Order Differential Equations

Second-order differential equations are differential equations of the form:

\[ \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b \]

If \( b = 0 \), we consider the differential equation to be homogeneous. Otherwise, the differential equation is said to be non-homogeneous.

1.1 Homogeneous Case

Let’s first consider the homogeneous case.

Recall the solution for a first-order homogeneous differential equation.

\[ \frac{dy}{dt} = \lambda y \]

Remember that differentiation is a linear operator, so this differential equation actually looks like an eigenvector/eigenvalue equation. Therefore, solving this equation is actually equivalent to finding an “eigenfunction” that corresponds to the eigenvalue \( \lambda \).

The solution is:

\[ y(t) = c e^{\lambda t}, c \in \mathbb{R} \]

In general, instead of solving a second order differential equation, we want to instead exploit what we know about first-order differential equations in order to find these “eigenfunctions” \( c e^{\lambda t} \).

\[ \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0 \]

First, we rewrite the differential equation as a matrix differential equation \( \frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \), where the state vector \( \vec{x}(t) \) is a vector-valued function.

For a second-order differential equation, we define \( \vec{x}(t) = \begin{bmatrix} y(t) \\ \frac{dy}{dt}(t) \end{bmatrix} \) and \( \frac{d}{dt} \vec{x}(t) = \begin{bmatrix} \frac{dy}{dt}(t) \\ \frac{d^2y}{dt^2}(t) \end{bmatrix} \), where \( \vec{x}(t) \) and \( \frac{d}{dt} \vec{x}(t) \) are 2-dimensional.

We then have the following matrix differential equation:

\[ \begin{bmatrix} \frac{dy}{dt}(t) \\ \frac{d^2y}{dt^2}(t) \end{bmatrix} = A \begin{bmatrix} y(t) \\ \frac{dy}{dt}(t) \end{bmatrix} \]

We can fill in the entries of \( A \) by rewriting the original differential equation \( \frac{d^2y}{dt^2} = -a_0 y - a_1 \frac{dy}{dt} \) for the second row of \( A \). For the first row of \( A \), we observe that \( \frac{dy}{dt}(t) \) appears in \( \vec{x}(t) \) and in its derivative \( \frac{d}{dt} \vec{x}(t) \).
Therefore,
\[
\begin{bmatrix}
\frac{dy}{dt}(t) \\
\frac{d^2 y}{dt^2}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-a_0 & -a_1
\end{bmatrix}
\begin{bmatrix}
y(t) \\
\frac{dy}{dt}(t)
\end{bmatrix}
\]

To transform this matrix differential equation into a system of first-order differential equations, we can diagonalize \(A\). Assuming that \(A = PDP^{-1}\) is diagonalizable, where \(D\) is a diagonal matrix with the eigenvalues of \(A\) and \(P\) is the matrix with the corresponding eigenvectors, we can rewrite our matrix differential equation.

\[
\frac{d}{dt} \vec{x}(t) = PDP^{-1}\vec{x}(t)
\]

\[
P^{-1} \frac{d}{dt} \vec{x}(t) = DP^{-1}\vec{x}(t)
\]

\[
\frac{d}{dt} P^{-1}\vec{x}(t) = DP^{-1}\vec{x}(t)
\]

We can then perform a change of variables. Let \(\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = P^{-1}\vec{x}(t)\).

\[
\frac{d}{dt} \vec{z}(t) = D\vec{z}(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \vec{z}(t)
\]

\[
\begin{bmatrix}
\frac{dz_1}{dt}(t) \\
\frac{dz_2}{dt}(t)
\end{bmatrix} =
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
z_1(t) \\
z_2(t)
\end{bmatrix}
\]

\[
\begin{cases}
\frac{dz_1}{dt}(t) = \lambda_1 z_1(t) \\
\frac{dz_2}{dt}(t) = \lambda_2 z_2(t)
\end{cases}
\]

We can now easily solve the system of first-order differential equations.

\[
z_1(t) = k_1 e^{\lambda_1 t} \quad z_2(t) = k_2 e^{\lambda_2 t}
\]

To find \(y(t)\), note that \(\vec{x}(t) = P\vec{z}(t)\).

\[
\begin{bmatrix}
y(t) \\
\frac{dy}{dt}(t)
\end{bmatrix} = P\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}
\]

Recall that \(P\) contains the eigenvectors of \(A\) and is so just a matrix of scalars. Therefore, \(y(t)\) is simply a linear combination of \(z_1(t)\) and \(z_2(t)\), which are in turn linear combinations of \(e^{\lambda_1 t}\) and \(e^{\lambda_2 t}\), respectively.

There are 3 cases for \(\lambda_1\) and \(\lambda_2\):

(a) \(\lambda_1\) and \(\lambda_2\) are both real. The solution is given by \(y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\).

(b) \(\lambda_1 = a + jb\) and \(\lambda_2 = a - jb\) are complex conjugates of each other. The general solution is given by \(y(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}\). However, using Euler's formula \(e^{j\theta} = \cos(\theta) + j\sin(\theta)\), we can eliminate the complex exponentials and rewrite the solution as \(y(t) = e^{at}(c_1 \cos(bt) + c_2 \sin(bt))\). The proof is left as an exercise for the reader.

(c) \(\lambda_1 = \lambda_2 = \lambda\). The solution is given by \(y(t) = c_1 e^{\lambda t} + c_2 te^{\lambda t}\). We will not prove this in class.

Simply stated, to solve a second-order differential equation, we need to find the eigenvalues of the \(A\) matrix, i.e., finding the roots of the characteristic equation and determine the general solution based on the three cases of \(\lambda_1\) and \(\lambda_2\).
1.2 Non-homogeneous Case

We can also solve non-homogeneous second-order differential equations where $b \neq 0$ is a constant term, i.e.

$$\frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b$$

We can still solve these equations using the method for homogeneous differential equations. If we substitute $\tilde{y} = y - \frac{b}{a_0}$, we note that $\frac{d\tilde{y}}{dt}(t) = \frac{dy}{dt}(t)$ and that $\frac{d^2\tilde{y}}{dt^2}(t) = \frac{d^2}{dt^2}(t)$.

$$\frac{d^2\tilde{y}}{dt^2}(t) + a_1 \frac{d\tilde{y}}{dt} + a_0 \tilde{y} = 0$$

Once we have solved for $\tilde{y}(t)$, we can reverse the substitution to get $y(t)$.

1.3 Initial Conditions

To solve for the scalars $c_1$ and $c_2$, we can set up a system of linear equations by plugging in the initial conditions into the general solution.

1. Solutions of Second-Order Differential Equations

Consider a differential equation of the form,

$$\frac{d^2 f}{dt^2}(t) + a_1 \frac{df}{dt}(t) + a_0 f(t) = 0,$$

such that

$$f(t) = c_1 e^{\lambda t} + c_2 e^{\lambda^* t},$$

where $f(\cdot)$ is a real valued function from $\mathbb{R}$ to $\mathbb{R}$ and $\lambda^*$ is the complex conjugate of $\lambda$.

(a) Use the fact that $f$ is real to prove that $c_1$ and $c_2$ are complex conjugates of each other.

Hint. Let $c_1 = a_1 + jb_1, c_2 = a_2 + jb_2$ and $\lambda = \sigma + j\omega$.

(b) Let $c_1 = a + jb, c_2 = a - jb$ and $\lambda = \sigma + j\omega$. Show that you can reduce $f(t)$ to the following form:

$$f(t) = (2a\cos(\omega t) - 2b\sin(\omega t)) e^{\sigma t}$$

(c) When solving for the original differential equation, why do we not need to solve for $c_1$ and $c_2$ and instead can directly jump to $a$ and $b$?

(d) Describe the behavior of $f(t)$ if $\sigma < 0$.

(e) Describe the behavior of $f(t)$ if $\sigma = 0$.

(f) Describe the behavior of $f(t)$ if $\sigma > 0$. 
2. Differential Equations

Solve the following second-order differential equations.

(a) \( \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 12 \), where \( y(0) = 1 \) and \( \frac{dy}{dt}(0) = 1 \)

(b) \( \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0 \), where \( y(0) = 2 \) and \( \frac{dy}{dt}(0) = -1 \)

(c) \( \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 13 \), where \( y(0) = 3 \) and \( \frac{dy}{dt}(0) = 7 \)

Contributors:

- Titan Yuan.