1. DFT Projection
   We can think about discrete sequences $x[n]$ with $N$ samples as $N$-dimensional vectors. In this $N$-dimensional vector space, the DFT is a coordinate transformation, and the DFT is computed by projecting these vectors onto a set of DFT basis vectors. The meaning of the coefficients on each of these basis vectors is the “frequency content” of the original signal $x[n]$.

   (a) The DFT basis vectors $\vec{u}_k$ are given by $u_k[n] = \frac{1}{N} e^{j\frac{2\pi}{N}kn}$ for $k = 0, 1, \ldots, N - 1$ and $n = 0, 1, \ldots, N - 1$. Write the basis vectors for sequences of length 2. Use these vectors to write the transformation matrix $F_N$ that fulfills the change of basis equation $\vec{X} = F_N \vec{x}$ where $\vec{X}$ is the representation of $\vec{x}$ in the frequency domain.

   (b) Plot these basis vectors as discrete sequences of length 2. Plot these basis vectors as vectors in the Cartesian plane $[x[0] \ x[1]]^\top$.

   (c) Find the basis vectors for sequences of length 3 and 4. Plot these basis vectors as discrete sequences (you can plot the imaginary component lighter, or as stems topped with an x instead of a dot). What is the length (magnitude) of the basis vectors? What property of a discrete sequence does the first basis vector correspond to in every case?

2. Roots of Unity
   The DFT is a coordinate transformation to a basis made up of roots of unity. In this problem, we explore some properties of the roots of unity. An $N$th root of unity is a complex number $z$ satisfying the equation $z^N = 1$ (or equivalently $z^N - 1 = 0$).

   (a) Show that $z^N - 1$ factors as $z^N - 1 = (z - 1) \left( \sum_{k=0}^{N-1} z^k \right)$.

   (b) Show that any complex number of the form $\omega_N^k = e^{j\frac{2\pi}{N}k}$ for $k \in \mathbb{Z}$ is an $N$th root of unity.

   (c) Draw the fifth roots of unity in the complex plane. How many unique fifth roots of unity are there?

   (d) Let $\omega_5 = e^{j\frac{2\pi}{5}}$. What is another expression for $\omega_5^{2^2}$?

   (e) What is the complex conjugate of $\omega_5$? What is the complex conjugate of $\omega_5^{2^2}$? What is the complex conjugate of $\omega_5^{2^3}$? Express all of your answers in terms of $\omega_5^k$ with $k = 0, 1, 2, 3, 4$.

   (f) Compute $\sum_{m=0}^{N-1} \omega_N^{km}$, where $\omega_N$ is an $N$th root of unity. Does the answer make sense in terms of the plot you drew?
(g) Write the expression for the \( N \) basis vectors \( \vec{u}_k \) of length \( N \) for the DFT of a signal of length \( N \) in terms of \( \omega_k \).

3. Denoising Signals using the DFT

Professor Maharbiz is sad. He just managed to create a beautiful audio clip consisting of a couple pure tones with beats and he wants Professor Roychowdhury to listen to it. He calls Professor Roychowdhury on a noisy phone and plays the message through the phone. Professor Roychowdhury then tells him that the audio is very noisy and that he is unable to truly appreciate the music. Unfortunately, Professor Maharbiz has no other means of letting Professor Roychowdhury listen to the message. Luckily, they have you! You propose to implement a denoiser at Professor Roychowdhury’s end.

(a) In the IPython notebook, listen to the noisy message. Plot the time signal and comment on visible structure, if any.

(b) Take the DFT of the signal and plot the magnitude. In a few sentences, describe what the spikes you see in the spectrum are.

*Hint:* Take a look at the documentation for `numpy.fft.fft`.

(c) There is a simple method to denoise this signal: simply threshold in the DFT domain! Threshold the DFT spectrum by keeping the coefficients whose absolute values lie above a certain value. Then take the inverse DFT and listen to the audio.

You will be given a range of possible values to test. Write the threshold value you think works best.

4. Aliasing

The concept of “aliasing” is intuitively about having a signal of interest whose samples look identical to a different signal of interest — creating an ambiguity as to which signal is actually present.

While the concept of aliasing is quite general, it is easiest to understand in the context of sinusoidal signals.

(a) Consider two signals

\[ x_1(t) = a \cos(2\pi f_0 t + \phi) \]

and

\[ x_2(t) = a \cos(2\pi (-f_0 + m f_s) t - \phi), \]

where \( f_s = \frac{1}{T_s} \). Are these two signals the same or different when viewed as functions of continuous time \( t \)?

(b) Consider the two signals from the previous part. These will both be sampled with the sampling interval \( T_s \). What will be the corresponding discrete-time signals \( x_{d,1}[n] \) and \( x_{d,2}[n] \)? (The \([n]\) refers to the \( n \)th sample taken — this is the sample taken at real time \( nT_s \).) Are they the same or different?

(c) What is the sinusoid \( a \cos(\omega t + \phi) \) that has the smallest \( \omega \geq 0 \) but still agrees at all of its samples (taken every \( T_s \) seconds) with \( x_1(t) \) from part(a)? Hint: Use solution from part b to find answer.

(d) Watch the following video: [https://www.youtube.com/watch?v=jQDjJRYmeWg](https://www.youtube.com/watch?v=jQDjJRYmeWg)

Assume the video camera running at 30 frames per second. That is to say, the camera takes 30 photos within a second, with the time between photos being constant.

Given that the main rotor has 5 blades, list *all* the possible rates at which the main rotor is spinning in revolutions per second assuming no physical limitations.

*Hint:* Your answer should depend on \( k \), where \( k \) can be any integer.
(e) Given that the back rotor has 3 blades and completes 2 revolutions in 1 second in the video, list all the possible rates at which the back rotor is spinning in revolutions per second assuming no physical limitations. Assume that the frame rate of the video is 30 frames per second (same as previous part).

*Hint:* Your answer should depend on $k$, where $k$ can be any integer.

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