1. DFT

(a) Compute the DFT coefficients of $x_1[n] = \cos\left(\frac{2\pi}{6} n\right)$ where $n \in \{0, 1, \ldots, 5\}$.

Solution:

$$\cos\left(\frac{2\pi}{6} n\right) = \frac{1}{2} e^{i\frac{2\pi}{6} n} + \frac{1}{2} e^{-i\frac{2\pi}{6} n}$$

$$u_k[n] = \frac{1}{6} e^{i\frac{2\pi}{6} kn}$$

$$\vec{x}_1 = \frac{6}{2} (\vec{u}_1 + \vec{u}_5)$$

$$X_1[k] = \begin{cases} 
3, & \text{if } k = 1, 5 \\
0, & \text{if } k \neq 1, 5 
\end{cases}$$

(b) Plot the time domain representation of $\vec{x}_1$. Plot the magnitude, $|X[n]|$, and plot the phase, $\angle X[n]$, for its DFT basis representation.

Solution:
(c) Compute the DFT coefficients of $x_2[n] = \cos\left(\frac{4\pi}{6}n\right)$ where $n \in \{0, 1, \ldots, 5\}$.

**Solution:**

\[
\hat{x}_2 = \frac{6}{2} (\bar{u}_2 + \bar{u}_4)
\]

\[
X_2[k] = \begin{cases} 
3, & \text{if } k = 2, 4 \\
0, & \text{if } k \neq 2, 4
\end{cases}
\]

(d) Plot the time-domain representation and the magnitude and phase for the DFT basis representation of $\hat{x}_2$.

**Solution:**
(e) How about the general case $x_p[n] = \cos\left(\frac{2\pi}{6}pn\right)$, where $n \in \{0, 1, \ldots, 5\}$?

Solution:

\[
\vec{x}_p = \frac{6}{2}(\vec{u}_p + \vec{u}_{6-p})
\]

\[
X_p[k] = \begin{cases} 
3, & \text{if } k = p, 6 - p \\
0, & \text{if } k \neq p, 6 - p 
\end{cases}
\]

(f) Compute the DFT coefficients of $\vec{s} = [1 
0 
1 
0 
1 
0]^T$.

Solution:

\[
\vec{s} = \frac{6}{2}(\vec{u}_0 + \vec{u}_3).
\]

\[
S[k] = \begin{cases} 
3, & \text{if } k = 0, 3 \\
0, & \text{if } k \neq 0, 3 
\end{cases}
\]

(g) Compute the DFT coefficients of $y_1[n] = \cos\left(\frac{2\pi}{6}n - \pi\right)$ where $n \in \{0, 1, \ldots, 5\}$.

Solution:

\[
\vec{y}_1 = \frac{6}{2}(-\vec{u}_1 - \vec{u}_5)
\]
\[ Y_1[k] = \begin{cases} -3, & \text{if } k = 1, 5 \\ 0, & \text{if } k \neq 1, 5 \end{cases} \]

2. LTI Filter

Suppose we apply the length \( L = 100 \) input

\[ u[n] = \cos(0.1\pi n) + \cos(0.4\pi n), \quad n = 0, 1, \ldots, 99 \]

to a finite impulse response filter whose impulse response is

\[ h[n] = \begin{cases} \frac{1}{6}, & n = 0, \ldots, 5 \\ 0, & \text{otherwise.} \end{cases} \]

We want to find the output \( y[n], n = 0, \ldots, 104 \) using the DFT.

(a) Find the 105-point DFT of \( u[n] \) by adding 5 zeroes to the length 100 signal given above and using the DFT command `numpy.fft.fft(x)`. Plot the magnitude \( |U[k]| \) against the frequency variable \( \omega \in [0, 2\pi] \). (Recall that the integer \( k \) corresponds to the frequency \( \frac{2\pi}{105} k \).)

**Solution:**

![DFT Plot](attachment:image.png)

(b) Find the 105-point DFT of \( h[n] \) by adding 99 zeroes to the length 6 impulse response given above. Plot the magnitude \( |H[k]| \) against the frequency variable \( \omega \in [0, 2\pi] \). Given this frequency response, how do you think each frequency component of \( u[n] \) will be affected when the filter is applied?

**Solution:**
This filter lets low frequencies pass while high frequencies are filtered out.

(c) Recall that the DFT coefficients of the output are

\[ Y[k] = H[k]U[k], \quad k = 0, \ldots, 104. \]

Find \( y[n] \) using the inverse DFT command \texttt{numpy.fft.ifft(x)}. Plot both \( u[n] \) and \( y[n] \) vs. time \( n \) and explain how the filter modified \( u[n] \).

\textbf{Solution:}

The high frequency oscillation has been attenuated by the filter.

\section*{3. DFT Sampling Matching}

For all parts, select the correct answer from the multiple choice options provided and justify your answer.

(a) A sampled time domain signal and its DFT coefficients are given below:
Now given the following time domain signal, which of the options below shows the correct DFT coefficient magnitudes?

Solution:
The correct DFT coefficients are shown in (B). The new signal completes 3 full cycles during the discrete sequence. Its amplitude is the same as the first sequence.

(b) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?
Solution:
The correct DFT coefficients are shown in (D). The DFT coefficients of a unit impulse are all 1. This impulse has been shifted, so the DFT coefficients have varying phase and are not purely real; however, their magnitudes are still uniformly 1.

(c) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?
**Solution:**
The correct DFT coefficients are shown in (C). The largest frequency component in this signal corresponds to 4 full cycles over the duration of the sequence. There is a smaller higher frequency component. The mean is 0.

**Contributors:**
- Yen-Sheng Ho.
- Harrison Wang.
- Murat Arcak.