1. Circuit Design

In this problem, you will find a circuit where several components have been left blank for you to fill in. Assume that the op-amp is ideal.

You have at your disposal only one of each of the following components (not including $R_1$ and $R_2$):

- (a) an open circuit
- (b) a short circuit
- (c) a resistor (you choose from the values $R = 1\, \text{k}\Omega, 15\, \text{k}\Omega, 30\, \text{k}\Omega$)
- (d) a capacitor (you choose from the values $C = 0.5\, \mu\text{F}, 1\, \mu\text{F}, 2\, \mu\text{F}$)

Consider the circuit below. The voltage source $V_{\text{in}}(t)$ has the form $V_{\text{in}}(t) = v_0 \cos(\omega t + \phi)$. The labeled voltages $V_{\text{in}}(\omega)$ and $V_{\text{out}}(\omega)$ are the phasor representations of $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$. The transfer function $H(\omega)$ is defined as $H(\omega) = \frac{V_{\text{out}}(\omega)}{V_{\text{in}}(\omega)}$.

(a) Let $R_1$ be $1\, \text{k}\Omega$. Fill in the boxes and determine the value of $R_2$, such that

- It is a high-pass filter.
- $|H(\infty)| = 10$.
- $|H(10^3)| = \sqrt{50}$.
• \(R_2\) must be one of the three values listed above.

(b) Draw the Bode plot of this transfer function.

2. Bandpass Filter

Consider the parallel bandpass filter below, where \(\widetilde{V}_s\) and \(\widetilde{V}_o\) are phasor voltages:

(a) What is the transfer function, \(H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_s}\), of this circuit in terms of \(R, L,\) and \(C\)?

(b) What is \(\omega_0\) of this filter?

(c) What is \(\omega_{c1}\) and \(\omega_{c2}\) of this filter? (Hint: \(H(\omega_{c1}) = H(\omega_{c2}) = \frac{1}{\sqrt{2}}\).)

(d) What is the bandwidth \(B\) of this filter?

(e) What is the \(Q\) of this filter?

3. Similarity Transforms

Consider the following circuit:

Recall that we constructed the following state space representation of this system.

\[
\begin{bmatrix}
\frac{dV_{C_2}}{dt} \\
\frac{d^2V_{C_2}}{dt^2} \\
\frac{dx}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 1 \\
-\left(\frac{1}{C_1C_2R_1R_2}\right) & -\left(\frac{C_1+C_2+C_2R_1}{C_1C_2R_2}\right) \\
0 & 0 & -\left(\frac{1}{C_1+C_2}\right)
\end{bmatrix} \begin{bmatrix}
V_{C_2} \\
\frac{dV_{C_2}}{dt} \\
x
\end{bmatrix}
\]
For simplicity, let \( R_1 = R_2 = R \) and \( C_1 = C_2 = C \). Then,

\[
\begin{bmatrix}
\frac{dV_{C_2}}{dt} \\
\frac{d^2V_{C_2}}{dt^2}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\left(\frac{1}{CR^2}\right) & -\left(\frac{3}{CR}\right)
\end{bmatrix}
\begin{bmatrix}
V_{C_2} \\
\frac{dV_{C_2}}{dt}
\end{bmatrix}
\]

We are going to try something different in this question: We’re going to instead use \( V_{C_1} \) and \( V_{C_2} \) as state variables and connect the two different state space representations.

(a) Find a system matrix, which we will denote \( \mathcal{A} \), such that,

\[
\begin{bmatrix}
\frac{dV_{C_1}}{dt} \\
\frac{dV_{C_2}}{dt}
\end{bmatrix} = \mathcal{A}
\begin{bmatrix}
V_{C_1} \\
V_{C_2}
\end{bmatrix}
\]

(b) Find a linear function that expresses \( \frac{dV_{C_2}}{dt} \) in terms of \( V_{C_1} \) and \( V_{C_2} \).

(c) Use the previous answers to construct a matrix \( T \) such that,

\[
\begin{bmatrix}
V_{C_2} \\
\frac{dV_{C_2}}{dt}
\end{bmatrix} = T
\begin{bmatrix}
V_{C_1} \\
V_{C_2}
\end{bmatrix}
\]

Succinctly,

\[ \vec{x} = T\vec{z} \]

(d) We know that

\[
\frac{d\vec{x}}{dt} = A\vec{x} \text{ and } \frac{d\vec{z}}{dt} = \mathcal{A}\vec{z}
\]

Use \( T \) from the previous question to conclude that

\[ \mathcal{A} = T^{-1}AT \]

(e) Let \( C = 1 \) and \( R = 1 \). Verify that,

\[ \mathcal{A} = T^{-1}AT \]

(f) Continuing the assumption that \( C = 1 \) and \( R = 1 \), find the eigenvalues of \( A \) and \( \mathcal{A} \). (Use a calculator.) What do you observe?

4. 1D Linear Approximations In Continuous Systems

Linearization is an incredible tool when it comes to studying systems with non-linear dynamics. (This is when system matrix \( A \) is dependent on the state variables.) We overcome this by fixing a point in state space, often denoted as \( x_0 \), and approximating the transitions about that point. To better understand this, we will work through a 1D example.

(a) Consider an arbitrary function of \( f(x) \) whose derivative \( \frac{df}{dx} \) is well defined. Construct a function of the form,

\[ g(x) = mx + b \]

that approximates \( f(x) \) in a neighborhood around a particular point \( x_0 \). \( m \) will be related to \( \frac{df}{dx} \).

*Hint:* Recall the definition of a derivative.
(b) We will study the following system.
\[ \frac{dx}{dt}(t) = f(x), \text{ where } f(x) = -2\sin\left(\frac{1}{3}x\right) \]

What is \( \frac{df}{dx}(x) \)?

(c) What are the equilibrium points for this system?

(d) Construct a linear approximation \( g(x) \) of \( f(x) \) about the point \( x_0 = 0 \).

(e) Using the above approximation, solve the system,
\[ \frac{dx}{dt}(t) = f(x) \approx g(x) \]
with \( x(0) = 1 \).

Note: This approximation is valid for points \( x \) close to \( x_0 \). We will explore this “closeness” when we study state feedback.

5. Spring and Mass

Let’s look at a mechanical spring-mass system governed by differential equations similar to those of electrical circuits.

Recall from physics that the motion of a mass is subject to Newton’s second law \( F = ma \) where \( a = \frac{dv}{dt} \) and \( v = \frac{dx}{dt} \) and that springs generate a force according to \( F_{sp} = -k\Delta x \), where \( k \) is the spring’s stiffness. We set \( x \) to be 0 when the spring is at its rest length \( l_0 \), so that \( \Delta x = x \). There is no gravity in this problem.

(a) Find a differential equation in terms of \( x \) and its derivatives that describes the motion of the mass. What order is this differential equation?

(b) Write the state space model for this system as \( \dot{x} = Ax \). What is your state vector?

(c) Find the eigenvalues of this system by solving \( \det(A - \lambda I) = 0 \). Is this system stable?

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