1. Circuit Design

In this problem, you will find a circuit where several components have been left blank for you to fill in. Assume that the op-amp is ideal.

You have at your disposal only one of each of the following components (not including $R_1$ and $R_2$):

- (a) an open circuit
- (b) a short circuit
- (c) a resistor (you choose from the values $R = 1\,\text{k}\Omega, 15\,\text{k}\Omega, 30\,\text{k}\Omega$)
- (d) a capacitor (you choose from the values $C = 0.5\,\mu\text{F}, 1\,\mu\text{F}, 2\,\mu\text{F}$)

Consider the circuit below. The voltage source $V_{\text{in}}(t)$ has the form $V_{\text{in}}(t) = v_0 \cos(\omega t + \phi)$. The labeled voltages $V_{\text{in}}(\omega)$ and $V_{\text{out}}(\omega)$ are the phasor representations of $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$. The transfer function $H(\omega)$ is defined as $H(\omega) = \frac{V_{\text{out}}(\omega)}{V_{\text{in}}(\omega)}$.

(a) Let $R_1$ be $1\,\text{k}\Omega$. Fill in the boxes and determine the value of $R_2$, such that

- It is a high-pass filter.
- $|H(\infty)| = 10$.
- $|H(10^3)| = \sqrt{50}$. 

This homework is due on Thursday, February 22, 2018, at 11:59AM (NOON). Self-grades are due on Monday, February 26, 2018, at 11:59AM (NOON).
• $R_2$ must be one of the three values listed above.

**Solution:**

Let the left box be $Z_1$ and the right box be $Z_2$. The circuit should be a high-pass filter, so $Z_2$ cannot be a short circuit or a capacitor (otherwise $V_{out}(\infty) = 0$). Since $Z_2$ cannot be a capacitor, $Z_1$ must be a capacitor. Otherwise, we would not have a zero or a pole in our circuit, which means it wouldn’t be a filter. $Z_2$ is either an open circuit or a resistor. Let $R_f = R_2 \parallel Z_2$ and $Z_1 = \frac{1}{j\omega C}$. The transfer function is given by

$$H(\omega) = -\frac{R_f}{R_1 + \frac{1}{j\omega C}} = -\frac{j\omega R_f C}{1 + j\omega R_1 C} = -\frac{R_f}{R_1} \frac{j\omega R_1 C}{1 + j\omega R_1 C}$$

Observing the transfer function, we know that $H(0) = 0$ and $H(\infty) = \frac{-R_f}{R_1}$, so it is a high-pass filter. From $|H(\infty)| = 10$, we know that $R_f = 10R_1 = 10k\Omega = R_2 \parallel Z_2$.

Since $10k\Omega$ is not an option for the resistor values, we know that $Z_2$ must be a resistor in order to get the correct gain. The equivalent resistance of two parallel resistors can never go higher than the smallest resistor in the parallel combination. This means $R_2$ and $Z_2$ cannot be $1k\Omega$. That leaves us with three options: both are $15k\Omega$, both are $30k\Omega$, or one is $30k\Omega$ and the other is $15k\Omega$. If you have two resistors with the same value in parallel, the equivalent resistance is half the original value. So if both were $15k\Omega$, the equivalent resistance would be $7.5k\Omega$, which is too low. For the case where they’re both $30k\Omega$, you get $15k\Omega$, which is too high. This means that the resistor pairing must be a $15k\Omega$ and $30k\Omega$ resistor. We can also double check using the parallel resistor equation:

$$R_1 \parallel R_2 = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

$$15k\Omega \parallel 30k\Omega = \frac{1}{\frac{1}{15k\Omega} + \frac{1}{30k\Omega}} = \frac{1}{\frac{1}{30k\Omega}} = 10k\Omega$$

To figure out the capacitance value for $Z_1$, we look at the boundary condition for $|H(10^3)| = \sqrt{50}$. Let $x = 10^3 R_1 C$.

$$\sqrt{50} = 10 \frac{\sqrt{x^2}}{\sqrt{1 + x^2}} \implies \frac{1}{2} = \frac{x^2}{1 + x^2} \implies x^2 = 1 \implies 10^3 R_1 C = 1 \implies C = \frac{1}{10^3 R_1} = 1\mu F$$

Thus,

• $R_2 = 15k\Omega$ (or $30k\Omega$).
• The right box: a $30k\Omega$ resistor (or $15k\Omega$ resistor if you said $R_2$ was $30k\Omega$).
• The left box: a capacitor with $C = 1\mu F$.

(b) Draw the Bode plot of this transfer function.

**Solution:**

Using the values found in part (a), we end up with the transfer function

$$H(\omega) = -\frac{R_f}{R_1} \frac{j\omega R_1 C}{1 + j\omega R_1 C} = -10 \frac{j\omega}{10^3}$$

This means that we have 1 zero at $0 \frac{rad}{s}$ and 1 pole at $10^3 \frac{rad}{s}$. We know that at high frequencies, the magnitude is 10, which is 20 dB. We have a negative sign in the DC gain, which contributes $-180^\circ$ to the phase at $0 \frac{rad}{s}$. We also have a zero at $0 \frac{rad}{s}$, which contributes $90^\circ$ at $0 \frac{rad}{s}$, which means that our initial phase is $-90^\circ$. 

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2. Bandpass Filter

Consider the parallel bandpass filter below, where \( \tilde{V}_s \) and \( \tilde{V}_o \) are phasor voltages:

(a) What is the transfer function, \( H(\omega) = \frac{\tilde{V}_o}{\tilde{V}_s} \), of this circuit in terms of \( R, L, \) and \( C \)?

Solution:

\[
Z_C = \frac{1}{j\omega C} \\
Z_L = j\omega L \\
Z_C \parallel Z_L = \frac{Z_C Z_L}{Z_C + Z_L} = \frac{j\omega L}{1 + (j\omega)^2 LC}
\]
\[ V_o = \frac{Z_C \parallel Z_L}{Z_C \parallel Z_L + R} \bar{V}_s = \frac{j \omega L}{1 + (j \omega)^2} \bar{V}_s = \frac{j \omega L}{(j \omega)^2 LCR + j \omega L + R} \]

\[ \bar{V}_s = \frac{j \omega L}{(j \omega)^2 LCR + j \omega L + R} \]

\[ H(\omega) = \frac{\bar{V}_o}{\bar{V}_s} = \frac{j \omega L}{(j \omega)^2 LCR + j \omega L + R} = \frac{j \omega L}{(j \omega)^2 LCR + j \omega L + R} \]

(b) What is \( \omega_0 \) of this filter?

**Solution:**

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ \omega_0 = 10^4 \text{rad} \text{s}^{-1} \]

(c) What is \( \omega_{c1} \) and \( \omega_{c2} \) of this filter? (Hint: \( |H(\omega_{c1})| = |H(\omega_{c2})| = \frac{1}{\sqrt{2}} \).)

**Solution:**

\[ \frac{1}{\sqrt{2}} = \frac{|j \omega L|}{(j \omega)^2 LCR + j \omega L + R} = \frac{\omega L}{\sqrt{(1 - \omega^2 LCR)^2 + \omega^2 L^2 R^2}} \]

Squaring both sides and cross-multiplying:

\[ \left(1 - \omega^2 LCR\right)^2 + \left(\frac{L}{R}\right)^2 \omega^2 = 2 \left(\frac{L}{R}\right)^2 \]

\[ \left(1 - \omega^2 LCR\right)^2 - \left(\frac{L}{R}\right)^2 = 0 \]

This is the difference of two squares, which we can factor as:

\[ \left(1 - \omega^2 LCR + \frac{L}{R}\right) \left(1 - \omega^2 LCR - \frac{L}{R}\right) = 0 \]

Using the quadratic formula:

\[ \omega_c = \frac{-L}{R} \pm \sqrt{\left(\frac{L}{R}\right)^2 + 4LC} \]

\[ \omega_c = \frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \]

Two of these frequencies will be negative (when the square root is negative), so ignore those answers:

\[ \omega_c = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \]

\[ \omega_{c1} = 9980 \text{ rad} \text{s}^{-1} \]

\[ \omega_{c2} = 10020 \text{ rad} \text{s}^{-1} \]
(d) What is the bandwidth $B$ of this filter?

**Solution:**

\[
B = \omega_c^2 - \omega_c^1 \\
B = 40 \text{ rad/s}
\]

(e) What is the $Q$ of this filter?

**Solution:**

\[
Q = \frac{\omega_0}{B} = \frac{10^4}{40} \\
Q = 250
\]

3. Similarity Transforms

Consider the following circuit.

Recall that we constructed the following state space representation of this system.

\[
\begin{bmatrix}
\frac{dV_C}{dt} \\
\frac{d^2V_C}{dt^2}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
-\left(\frac{1}{C_1C_2R_1R_2}\right) & -\left(\frac{C_1+C_2+\frac{R_2}{R_1}}{C_1C_2R_1R_2}\right)
\end{bmatrix}
\begin{bmatrix}
V_C \\
\frac{dV_C}{dt}
\end{bmatrix}
\]

For simplicity, let $R_1 = R_2 = R$ and $C_1 = C_2 = C$. Then,

\[
\begin{bmatrix}
\frac{dV_C}{dt} \\
\frac{d^2V_C}{dt^2}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
-\left(\frac{1}{CR}\right) & -\left(\frac{1}{CR}\right)
\end{bmatrix}
\begin{bmatrix}
V_C \\
\frac{dV_C}{dt}
\end{bmatrix}
\]

We are going to try something different in this question: We're going to instead use $V_{C1}$ and $V_{C2}$ as state variables and connect the two different state space representations.
(a) Find a system matrix, which we will denote \( A \), such that,

\[
\begin{bmatrix}
\frac{dV_{C_1}}{dt} \\
\frac{dV_{C_2}}{dt} \\
\frac{dz}{dt}
\end{bmatrix} = A 
\begin{bmatrix}
V_{C_1} \\
V_{C_2} \\
\vec{z}
\end{bmatrix}
\]

**Solution:**

Observe that

\[ i_1 = \frac{V_{C_1}}{R_1}, \quad i_2 = C_1 \frac{dV_{C_1}}{dt}, \quad i_3 = C_2 \frac{dV_{C_2}}{dt} \]

and that

\[ V_{C_1} = i_3 R_2 + V_{C_2}. \]

This gives us our first equation.

\[
\frac{dV_{C_2}}{dt} = \frac{V_{C_1}}{C_2 R_2} - \frac{V_{C_2}}{C_2 R_2}
\]

From KCL, we have,

\[ i_1 + i_2 + i_3 = 0 \]

or, after plugging in the expression for \( \frac{dV_{C_2}}{dt} \),

\[
\frac{V_{C_1}}{R_1} + C_1 \frac{dV_{C_1}}{dt} + \frac{V_{C_1}}{R_2} - \frac{V_{C_2}}{R_2} = 0
\]

This reduces to our second equation.

\[
\frac{dV_{C_1}}{dt} = \frac{V_{C_2}}{C_1 R_2} - V_{C_1} \left( \frac{1}{C_1 R_1} + \frac{1}{C_1 R_2} \right)
\]

Combining the two, we get

\[
\begin{bmatrix}
\frac{dV_{C_1}}{dt} \\
\frac{dV_{C_2}}{dt} \\
\frac{dz}{dt}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{R_1 R_2} & \frac{1}{C_1 R_2} & \frac{1}{C_1 R_1} \\
\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_1} & -\frac{1}{C_2 R_2}
\end{bmatrix} 
\begin{bmatrix}
V_{C_1} \\
V_{C_2} \\
\vec{z}
\end{bmatrix}
\]

Using \( R_1 = R_2 = R \) and \( C_1 = C_2 = C \),

\[
\begin{bmatrix}
\frac{dV_{C_1}}{dt} \\
\frac{dV_{C_2}}{dt} \\
\frac{dz}{dt}
\end{bmatrix} = \begin{bmatrix}
\frac{2R}{CR} & \frac{R}{CR} & \frac{1}{CR} \\
\frac{R}{CR} & -\frac{1}{CR} & -\frac{R}{CR}
\end{bmatrix} 
\begin{bmatrix}
V_{C_1} \\
V_{C_2} \\
\vec{z}
\end{bmatrix}
\]

(b) Find a linear function that expresses \( \frac{dV_{C_2}}{dt} \) in terms of \( V_{C_1} \) and \( V_{C_2} \).

**Solution:**

We get this from the previous problem.

\[ \frac{dV_{C_2}}{dt} = \frac{V_{C_1}}{C_2 R_2} - \frac{V_{C_2}}{C_2 R_2} \]

Using \( R_1 = R_2 = R \) and \( C_1 = C_2 = C \),

\[ \frac{dV_{C_2}}{dt} = \frac{V_{C_1}}{CR} - \frac{V_{C_2}}{CR} \]
(c) **OPTIONAL**: Use the previous answers to construct a matrix $T$ such that,

$$
\begin{bmatrix}
V_{C_2} \\
\frac{dV_{C_2}}{dt}
\end{bmatrix} = T \begin{bmatrix}
V_{C_1} \\
V_{C_2}
\end{bmatrix}
$$

Succinctly,

$$\vec{x} = T \vec{z}$$

**Solution:**

$$
\begin{bmatrix}
V_{C_2} \\
\frac{dV_{C_2}}{dt}
\end{bmatrix} = T \begin{bmatrix}
0 & 1 \\
\frac{1}{CR} & -\frac{1}{CR}
\end{bmatrix} \begin{bmatrix}
V_{C_1} \\
V_{C_2}
\end{bmatrix}
$$

(d) **OPTIONAL**: We know that

$$\frac{d\vec{x}}{dt} = A\vec{x} \text{ and } \frac{d\vec{z}}{dt} = \mathcal{A}\vec{z}$$

Use $T$ from the previous question to conclude that

$$\mathcal{A} = T^{-1}AT$$

**Solution:**

We have

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

Observe that,

$$\frac{dT\vec{z}}{dt} = AT\vec{z}$$

Since $\frac{d\vec{z}}{dt}$ is linear with respect to the state variables,

$$T\frac{d\vec{z}}{dt} = AT\vec{z}$$

Thus,

$$\frac{d\vec{z}}{dt} = T^{-1}AT\vec{z}$$

Since,

$$\frac{d\vec{z}}{dt} = \mathcal{A}\vec{z}$$

we can conclude that,

$$\mathcal{A} = T^{-1}AT$$

(e) **OPTIONAL**: Let $C = 1$ and $R = 1$. Verify that,

$$\mathcal{A} = T^{-1}AT$$

**Solution:**

$$T^{-1} = \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix}^{-1} = \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}$$

$$T^{-1}AT = \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
-1 & -3
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix} = \begin{bmatrix}
-1 & -2 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix} = \begin{bmatrix}
-2 & 1 \\
1 & -1
\end{bmatrix} = \mathcal{A}$$
(f) **OPTIONAL**: Continuing the assumption that $C = 1$ and $R = 1$, find the eigenvalues of $A$ and $\mathcal{A}$. (Use a calculator.) What do you observe?

**Solution:**
They both have the same eigenvalues: $\lambda = -0.38196601, -2.61803399$.

### 4. 1D Linear Approximations In Continuous Systems

Linearization is an incredible tool when it comes to studying systems with non-linear dynamics. (This is when system matrix $A$ is dependent on the state variables.) We overcome this by fixing a point in state space, often denoted as $x_0$, and approximating the transitions about that point. To better understand this, we will work through a 1D example.

(a) Consider an arbitrary function of $f(x)$ whose derivative $\frac{df}{dx}$ is well defined. Construct a function of the form,

$$g(x) = mx + b$$

that approximates $f(x)$ in a neighborhood around a particular point $x_0$. $m$ will be related to $\frac{df}{dx}$.

*Hint:* Recall the definition of a derivative.

**Solution:**
The definition of a derivative is:

$$\frac{df}{dx}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This tells us that,

$$f(x) \approx \frac{df}{dx}(x_0)(x - x_0) + f(x_0)$$

or,

$$f(x) \approx \frac{df}{dx}(x_0)x + \left( f(x_0) - \frac{df}{dx}(x_0)x_0 \right)$$

(b) We will study the following system.

$$\frac{dx}{dt}(t) = f(x), \text{ where } f(x) = -2\sin\left(\frac{1}{3}x\right)$$

What is $\frac{df}{dx}(x)$?

**Solution:**

$$\frac{df}{dx} = -\frac{2}{3}\cos\left(\frac{1}{3}x\right)$$

(c) What are the equilibrium points for this system?

**Solution:**

$$x^* = 3\pi n, n \in \mathbb{Z}$$
(d) Construct a linear approximation $g(x)$ of $f(x)$ about the point $x_0 = 0$.

**Solution:**

$$f(x) \approx -\frac{2}{3}x$$

(e) Using the above approximation, solve the system,

$$\frac{dx}{dt}(t) = f(x) \approx g(x)$$

with $x(0) = 1$.

**Solution:**

$$\frac{dx}{dt} \approx -\frac{2}{3}x(t) \implies x(t) \approx e^{-\frac{2}{3}t}$$

**Note:** This approximation is valid for points $x$ close to $x_0$. We will explore this “closeness” when we study state feedback.

5. Spring and Mass

Let’s look at a mechanical spring-mass system governed by differential equations similar to those of electrical circuits.

Recall from physics that the motion of a mass is subject to Newton’s second law $F = ma$ where $a = \frac{dv}{dt}$ and $v = \frac{dx}{dt}$ and that springs generate a force according to $F_{sp} = -k\Delta x$, where $k$ is the spring’s stiffness. We set $x$ to be 0 when the spring is at its rest length $l_0$, so that $\Delta x = x$. There is no gravity in this problem.

(a) Find a differential equation in terms of $x$ and its derivatives that describes the motion of the mass. What order is this differential equation?

**Solution:**

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This is a second-order differential equation.

(b) Write the state space model for this system as $\dot{\bar{x}} = A\bar{x}$. What is your state vector?

**Solution:**

For the state vector $\bar{x} = \begin{bmatrix} x \\ v \end{bmatrix}$, we can write:

$$\dot{\bar{x}} = \begin{bmatrix} x \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$
If you flipped \( x \) and \( v \) in your state vector, your \( A \) matrix will be transposed.

(c) Find the eigenvalues of this system by solving \( \det(A - \lambda I) = 0 \). Is this system stable?

**Solution:**

\[
\lambda^2 + \frac{k}{m} = 0
\]

\[
\lambda_1 = \sqrt{\frac{k}{m}} j
\]

\[
\lambda_2 = -\sqrt{\frac{k}{m}} j
\]

Both eigenvalues are strictly imaginary. This means that the system will oscillate forever without damping out or blowing up and that the system is marginally stable.

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