1. Otto the Pilot

Otto has devised a control algorithm, so that his plane climbs to the desired altitude by itself. However, he is having oscillatory transients as shown in the figure. Prof. Roychowdchury told him that if his system has complex eigenvalues

\[ \lambda_{1,2} = v \pm j\omega, \]

then his altitude would indeed oscillate with frequency \( \omega \) about the steady state value, 1 km, and that the time trace of his altitude would be tangent to the curves \( 1 + e^{vt} \) and \( 1 - e^{vt} \) near its maxima and minima respectively.

(a) Find the real part \( v \) and the imaginary part \( \omega \) from the altitude plot.

**Solution:**

Solving \( 1 + e^{5v} = 1.4843 \) gives us \( v = -0.1450 \, \text{min}^{-1} \). Then, comparing the maxima that are separated by an interval of 10 minutes gives \( \omega = \frac{2\pi}{10} = 0.62832 \, \text{rad min}^{-1} \).

If you solved in units of \( \frac{1}{\text{min}} \) and \( \text{rad s}^{-1} \), then \( v = -0.0024 \, \text{min}^{-1} \) and \( \omega = 0.0105 \, \text{rad s}^{-1} \).

(b) Let the dynamical model for the altitude be

\[
\frac{d}{dt} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},
\]

where \( y(t) \) is the deviation of the altitude from the steady state value, \( \dot{y}(t) \) is the time derivative of \( y(t) \), and \( a_1 \) and \( a_2 \) are constants. Using your answer to part (a), find what \( a_1 \) and \( a_2 \) are.

**Solution:**
The eigenvalues of \( A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \) are given by \( 0 = \lambda^2 - a_2 \lambda - a_1 \), or equivalently,

\[
\lambda = \frac{a_2 \pm \sqrt{a_2^2 + 4a_1}}{2} = v \pm j\omega.
\]

Solving for \( a_1 \) and \( a_2 \) (using the \( \frac{1}{\text{min}} \) and \( \frac{\text{rad}}{\text{min}} \) values of \( v \) and \( \omega \)), we get

\[
a_2 = 2v = -0.2900 \quad \text{and} \quad a_1 = -\omega^2 - \frac{a_2^2}{4} = -0.4158.
\]

If you solved using the \( \frac{1}{\text{s}} \) and \( \frac{\text{rad}}{\text{s}} \) values of \( v \) and \( \omega \), then

\[
a_2 = 2v = -0.0048 \quad \text{and} \quad a_1 = -\omega^2 - \frac{a_2^2}{4} = -1.16 \cdot 10^{-4}.
\]

(c) Otto can change \( a_2 \) by turning a knob. Tell him what value he should pick so that he has a “critically damped” ascent with two real negative eigenvalues at the same location.

**Solution:**

To get two real identical eigenvalues, Otto should choose \( a_2 \) to make \( a_2^2 + 4a_1 = 0 \). This means that \( a_2 = \pm 2\sqrt{-a_1} \). Since \( a_2 \) must be negative for the system to be stable, we only look at the negative root.

Solving with the \( a_1 \) derived from the \( \frac{1}{\text{min}} \) and \( \frac{\text{rad}}{\text{min}} \) values of \( v \) and \( \omega \), he should tune his knob to

\[
a_2 = -2\sqrt{-a_1} = -2\sqrt{0.4158} = -1.2897.
\]

If you solved using \( a_1 \) derived from the \( \frac{1}{\text{s}} \) and \( \frac{\text{rad}}{\text{s}} \) values of \( v \) and \( \omega \), then you get

\[
a_2 = -2\sqrt{-a_1} = -2\sqrt{1.16 \cdot 10^{-4}} = -0.0215.
\]

2. **LED Strip**

I have an LED strip with 5 red LEDs whose brightnesses I want to set. These LEDs are addressed as a queue: at each time step, I can push a new brightness command between 0 and 255 to the left-most LED. Each of the following LEDs will then take on the brightness previously displayed by the LED immediately to its left.

(a) What should we use for our state vector? What does it mean that this is a state vector? What is our input?

**Solution:**

We can use the brightnesses of each LED as our state vector. We can use these values as our state vector since together with the input, they describe everything about our system that we need to know in order to predict what our system will do in the future. Our input is the command to the left-most LED.
(b) Is our system linear? If it is linear, write out the state equations in matrix form. Please choose a reasonable order for the state variables in the state vector.

**Solution:**
The system is linear because it can be written in the form $\vec{x}[t+1] = A\vec{x}[t] + Bu[t]$. Ordering the LED brightnesses in the state vector from left to right, we get:

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u[t]$$

If you chose to put the left-most LED’s brightness last in the state vector (so that the LEDs are ordered right to left and the state vector gets flipped upside down), the A matrix gets transposed and the B matrix is flipped upside down.

(c) Is this system controllable? Explain intuitively what this system’s controllability means in terms of the LED brightnesses.

**Solution:**
Testing for controllability, we have:

$$[A^4 B \ A^3 B \ A^2 B \ AB \ B] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which has full rank. This means that the system is controllable. A system is called controllable if from any initial state, we can reach any final state that we desire at some time in the future.

For our LED strip, controllability means that we can display any set of brightnesses that we desire, but it may take a few time steps to get there.

(d) Is this system stable?

**Solution:**
The eigenvalues of this discrete-time system are all 0, which is inside the unit circle. Therefore, the system is stable.

(e) Starting from the pattern of brightnesses (from left to right) $[0 \ 127 \ 0 \ 255 \ 0]$, can we maintain this pattern for all future time steps? Can we display any fixed pattern of brightnesses for all time?

**Solution:**
We cannot display $[0 \ 127 \ 0 \ 255 \ 0]$ for all time time. Immediately after we display this set of brightnesses, we will display $[u[1] \ 0 \ 127 \ 0 \ 255]$. If we want to display a fixed and unchanging set of brightnesses, every element in our state vector must be the same.

Controllability tells us only that we can reach any desired state (sometimes only temporarily). It does not mean we can keep our system at any desired state for all time.
3. Understanding the SIXT33N Car Control Model

As we continue along the process of making the SIXT33N cars be awesome, we’d like to better understand the car model that we will be using to develop a control scheme. As a wheel on the car turns, there is an encoder disc (see below) that also turns as the wheel turns. The encoder shines a light through the encoder disc, and as the wheel turns, the light is continually blocked and unblocked, allowing the encoder to detect rotations of the wheel per ‘tick’ of the encoder disc.

The following model applies separately to each motor of the car:

\[ v[t] = d[t+1] - d[t] = \theta u[t] - \beta \]

Meet the variables at play in this model:

- \( t \) - The current timestep of the model. Since we model the car as a discrete system, this will advance by 1 on every new sample in the system.
- \( d[t] \) - The current number of ticks advanced by this wheel.
- \( v[t] \) - The discrete-time velocity (in units of ticks/timestep) of the wheel, measured by finding the difference between two adjacent tick counts \((d[t+1] - d[t])\).
- \( u[t] \) - The input to the system, in terms of “PWMs.” As we will observe in lab, the circuit driving the model controls the amount of power delivered to the wheel in units of PWM intensity. This is a number between 0 and 255, where 0 means that no power is delivered to the motor and 255 means that maximum capable power is delivered to the motor.
- \( \theta \) - In units of “ticks/PWM,” this models how much more the motor turns for every increase in PWM. This is empirically measured from the car.
- \( \beta \) - In units of ticks/timestep, this models the effect of friction on the car (if no power is applied to the motors, we’d expect the velocities of the motors to decrease). This is empirically measured from the car.

In this problem, we will assume that the motor conforms perfectly to this model to get an intuition of how the model works.

(a) If we wanted to make the motor drive at a certain target velocity \( v^* \), with what PWM \( u[t] \) should we feed the motor?

**Solution:**
\[ v^* = \theta u[t] - \beta \]
\[ v^* + \beta = \theta u[t] \]
\[ u[t] = \frac{v^* + \beta}{\theta} \]

(b) What signs should \( \theta \) and \( \beta \) have? Should they be positive or negative? Note that applying PWM to the motor driver circuit can only ever deliver power in a way so as to cause the motor to move forwards and never backwards and that there are no braking mechanisms on the motor.

**Solution:**
\( \theta > 0 \) since applying PWM should only ever increase the velocity. If friction is a large factor in the linear fit for our motor model, we expect that \( \beta \) should be greater than zero since friction should decrease the speed of the car in the absence of PWM input. However, nonlinearities and imperfections in our motors may outweigh the effect of friction, so that \( \beta \) may experimentally end up positive.

(c) Even if the motor conforms perfectly to the model, our inputs still limit the range of velocities of the motor. Given that \( 0 \leq u[t] \leq 255 \) \(^1\) determine the maximum and minimum velocities possible with the motor. What does this tell us about the braking of the car?

**Solution:**
The maximum is \( 255 \theta - \beta \), and the minimum is \( 0 - \beta = -\beta \).
Since there are no brakes on the motor, we slow down by reducing the PWM.

(d) Our intuition tells us that a motor on a car should eventually stop turning if we stop applying any power to it. Find \( v(t) \) as \( t \to \infty \), assuming that \( v(0) = v_0 \) (say \( v_0 > 0 \)) and \( u[t] = 0 \). Does our model obey our intuition? What does that tell us about our model?

**Solution:**
\[ v(\infty) = -\beta \]

However, our intuition says that the motor should have stopped: \( v(\infty) = 0 \). In lab, we will empirically find the value of \( \beta \) over a range of PWM values, but our fit does not work very well everywhere and our model does not match the real behavior near \( u = 0 \).

(e) In order to characterize the car, we need to find the \( \theta \) and \( \beta \) values that model your left and right motors: \( \theta_l, \beta_l, \theta_r, \) and \( \beta_r \). We start by running our car on the ground to collect data. From the data collection, we get the distance covered by each motor, from which we can calculate the velocity of each motor over time (already done for you in the attached IPython notebook). We also know our inputs to each motor at each time step. Using the attached IPython notebook, find the best estimate for \( \theta_l, \beta_l, \theta_r, \) and \( \beta_r \) using least squares linear regression.

**Solution:**

\(^1\)See [https://www.arduino.cc/en/uploads/Tutorial/pwm1.gif](https://www.arduino.cc/en/uploads/Tutorial/pwm1.gif) for an example of how PWM works and why this is the case.
Using the least squares linear regression method shown below, we can estimate the slope and y-intercept for each graph.

\[
\text{Slope} = \frac{N \sum(xy) - \sum x \sum y}{N(\sum x^2) - (\sum x)^2}
\]

\[
\text{y-intercept} = \frac{\sum y - \text{Slope} \times (\sum x)}{N}
\]

where \(N\) is the number of points. In our case, \(x, y, N\) are \(u, v_{\text{left}}\) or \(v_{\text{right}}\), and 31, respectively. Therefore, the estimated values are as follows:

\[
\theta_{\text{left}} = 0.4449
\]

\[
\theta_{\text{right}} = 0.4669
\]

\[
\beta_{\text{left}} = -5.3172
\]

\[
\beta_{\text{right}} = 10.2654
\]

(f) Now, using the \(\theta\) and \(\beta\) values you found above, plot the predicted velocities of the left and right motor over the inputs \(u\) from 0 to 255.

**Solution:**

We can simply plot the two equations:

\[
v_{\text{left}} = \theta_{\text{left}} u - \beta_{\text{left}}
\]

\[
v_{\text{right}} = \theta_{\text{right}} u - \beta_{\text{right}}
\]
Therefore, the measured data and the predicted curves match pretty well for both motors.

4. Redo problem 1 of the midterm.

(a)

(b)

(c)

5. Redo problem 2 of the midterm.

(a)

(b)

(c)

(d)

6. Redo problem 3 of the midterm.

(a)

(b)

7. Redo problem 4 of the midterm.

(a)

(b)

(c)

8. Redo problem 5 of the midterm.
(a)
(b)
(c)
(d)
EE 16B Midterm 1
Spring 2018

Name:_____________________________________________________

SID #:______________________________________________________

(after the exam begins add your SID# in the top right corner of each page)

Discussion Section and TA:_____________________________
Discussion Section and TA:_____________________________
Lab Section and TA:___________________________________

Name of left neighbor:_________________________________
Name of right neighbor:________________________________

Instructions:

Show your work. An answer without explanation is not acceptable and does not guarantee any credit.

Only the front pages will be scanned and graded. Back pages won't be scanned; you can use them as scratch paper.

Do not remove pages, as this disrupts the scanning. If needed, cross out any parts that you don't want us to grade.

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>MAX</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
</tbody>
</table>
Problem 1 Warm up (15 points)
a) Consider the following circuit.

\[ \frac{d^2 v_{out}}{dt^2} + A \frac{dv_{out}}{dt} + B v_{out} = 0 \]

For \( t \geq 0 \), the following equation applies to \( v_{out}(t) \). In addition, \( v_{out}(0) = V_0 \) and \( \frac{dv_{out}}{dt} = 0 \) at \( t = 0 \).

If \( A < 2\sqrt{B} \), provide an expression for \( v_{out}(t) \geq 0 \). (5 points)

Solution:

\[ v_{out}(t) = \]
b) Consider the circuit below.

What is \( \tilde{H}_{\text{out}}(\omega) = \frac{\tilde{V}_{\text{out}}(\omega)}{\tilde{V}_{\text{in}}(\omega)} \) for \( \omega \to \infty \)? (5 points)

Solution:

\( \tilde{H}_{\text{out}}(\omega \to \infty) = \)
c) Consider the Bode plot below. (5 points)

This is a Bode magnitude plot of the transfer function $\tilde{H}(\omega)$. The expression for $\tilde{H}(\omega)$ is shown below.

$\tilde{H}(\omega) = \frac{\tilde{H}_x(\omega)}{1 + j \left( \frac{\omega}{\omega_c} \right)}$

What is $\tilde{H}_x(\omega)$?

Solution:

$\tilde{H}_x(\omega) =$
Problem 2 H’s and Bodes… (25 points)

Consider the circuit below. There is nothing connected to the $V_{out}$ terminal. (5 points)

\[ H_{out}(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} \]

Solution:

\[ \tilde{H}_{out}(\omega) = \]
b) For this part of the problem, assume you have ONE capacitor, ONE inductor and ONE resistor. If \( Z_2 = 0 \) for all \( \omega \), which components would you choose for \( Z_1 \) and \( Z_3 \) such that the filter response is a passive low pass filter with a slope of -20 dB/decade for frequencies beyond a single cutoff frequency? (5 points)

**Solution:**

<table>
<thead>
<tr>
<th>Circle ONE component to go into the ( Z_1 ) box:</th>
<th>Capacitor</th>
<th>Inductor</th>
<th>Resistor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle ONE component to go into the ( Z_3 ) box:</td>
<td>Capacitor</td>
<td>Inductor</td>
<td>Resistor</td>
</tr>
</tbody>
</table>
c) Consider the following circuit:

We define a transfer function $\tilde{H}_{\text{out}}(\omega) = \frac{I_{\text{out}}(\omega)}{V_{\text{in}}(\omega)}$.

**LOOK AT THE DEFINITION OF THE TRANSFER FUNCTION CAREFULLY.**

Provide an expression in *canonical form* for $\tilde{H}_{\text{out}}(\omega)$. (10 points)

Solution:

$$\tilde{H}_{\text{out}}(\omega) =$$
d) If $L = 1\ \text{H}$ and $R_1 = R_2 = 1\ \Omega$, provide below magnitude and phase Bode plots for $\tilde{H}_{\text{out}}(\omega)$. (5 points)
Problem 3  Transistors and RC’s (15 points)
Consider the circuit below.

a) Fill in the truth table below for the circuit above. \( V_A, V_B \) and \( V_{out} \) are digital voltages that can only assume values of 0 or \( V_{DD} \). (5 points)

<table>
<thead>
<tr>
<th>( V_A )</th>
<th>( V_B )</th>
<th>( V_{out} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( V_{DD} )</td>
<td></td>
</tr>
<tr>
<td>( V_{DD} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( V_{DD} )</td>
<td>( V_{DD} )</td>
<td></td>
</tr>
</tbody>
</table>
For this part, assume that $V_A = V_B = V_{DD}$ for $-\infty < t < 0$. At $t=0$, $V_A$ and $V_B$ switch instantly from $V_{DD}$ to 0. Assume all transistors behave as resistors with the same value, $R$, if in the ON state and that all capacitances are already accounted for in the circuit above.

b) Provide an expression for $V_{out}(t \geq 0)$. (10 points)
Problem 4 Phasors! (20 points)
Consider the circuit below.

We are going to solve this circuit, which contains both a sinusoidal and a DC source using superposition and phasors.

a) Solve for $v_{\text{OUT}}(t)$ if $i_{\text{IN}}(t) = 0$ and $V_{\text{DD}}$ = a non-zero constant. (5 points)

Solution:
b) Solve for \( v_{\text{OUT}}(t) \) if \( i_{\text{IN}}(t) = I_0 \cos(\omega t) \) and \( V_{\text{DD}} = 0 \text{ V.} \) (10 points)
c) Solve for $v_{\text{OUT}}(t)$ if $i_{\text{IN}}(t) = I_0 \cos(\omega t)$ and $V_{\text{DD}} = \text{a non-zero constant}$. (5 points)

Solution:
Problem 5 (25 points)
a) Consider the following circuit. The switch is closed for $t<0$, then opens at $t=0$. Both of the independent sources have a DC value (i.e. they do not change with time).

**Solution:**

b) What is $v_Y(t<0)$? (5 points)
c) Provide an equation in the variable $i_x(t)$ that, when solved, would provide an expression for $i_x(t)$ for $t \geq 0$. DO NOT SOLVE THE EQUATION. (10 points)

Solution:

\[
\text{d) If } C_x = C_y = C = 1 \text{ F and } R_x = R = 1 \text{ } \Omega \text{ and } L_y = L = 1 \text{ } \text{H, provide an expression for } i_x(t) \text{ for } t \geq 0. \text{ (5 points) }
\]

Solution:

\[
i_x(t) =
\]
<table>
<thead>
<tr>
<th>Factor</th>
<th>Bode Magnitude</th>
<th>Bode Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Constant</strong> ( K )</td>
<td>20 ( \log K )</td>
<td>( \pm 180^\circ ) if ( K &lt; 0 )</td>
</tr>
<tr>
<td></td>
<td>0 dB</td>
<td>0°</td>
</tr>
<tr>
<td></td>
<td>( \omega )</td>
<td>( \omega )</td>
</tr>
<tr>
<td><strong>Zero @ Origin</strong></td>
<td>slope = 20N dB/decade</td>
<td>( (90N)^\circ )</td>
</tr>
<tr>
<td>( (j\omega)^N )</td>
<td>0°</td>
<td>( \omega )</td>
</tr>
<tr>
<td></td>
<td>( \omega )</td>
<td>( \omega )</td>
</tr>
<tr>
<td><strong>Pole @ Origin</strong></td>
<td>slope = -20N dB/decade</td>
<td>( (-90N)^\circ )</td>
</tr>
<tr>
<td>( (j\omega)^{-N} )</td>
<td>0°</td>
<td>( \omega )</td>
</tr>
<tr>
<td></td>
<td>( \omega )</td>
<td>( \omega )</td>
</tr>
<tr>
<td><strong>Simple Zero</strong></td>
<td>slope = 20N dB/decade</td>
<td>( (90N)^\circ )</td>
</tr>
<tr>
<td>( (1 + j\omega/\omega_c)^N )</td>
<td>0°</td>
<td>( \omega )</td>
</tr>
<tr>
<td></td>
<td>( \omega )</td>
<td>( \omega )</td>
</tr>
<tr>
<td><strong>Simple Pole</strong></td>
<td>slope = -20N dB/decade</td>
<td>( (-90N)^\circ )</td>
</tr>
<tr>
<td>( \left( \frac{1}{1 + j\omega/\omega_c} \right)^N )</td>
<td>0°</td>
<td>( \omega )</td>
</tr>
<tr>
<td></td>
<td>( \omega )</td>
<td>( \omega )</td>
</tr>
<tr>
<td><strong>Quadratic Zero</strong></td>
<td>slope = 40N dB/decade</td>
<td>( (180N)^\circ )</td>
</tr>
<tr>
<td>( (1 + j2\xi\omega/\omega_c + (j\omega/\omega_c)^2)^N )</td>
<td>0°</td>
<td>( \omega )</td>
</tr>
<tr>
<td></td>
<td>( \omega )</td>
<td>( \omega )</td>
</tr>
<tr>
<td><strong>Quadratic Pole</strong></td>
<td>slope = -40N dB/decade</td>
<td>( (-180N)^\circ )</td>
</tr>
<tr>
<td>( \left[1 + j2\xi\omega/\omega_c + (j\omega/\omega_c)^2 \right]^N )</td>
<td>0°</td>
<td>( \omega )</td>
</tr>
<tr>
<td></td>
<td>( \omega )</td>
<td>( \omega )</td>
</tr>
</tbody>
</table>
Solution:
EE 16B Midterm 1
Spring 2018

Name: _________________________________

SID #: ________________________________
(after the exam begins add your SID# in the top right corner of each page)

Discussion Section and TA: __________________________
Discussion Section and TA: __________________________
Lab Section and TA: _________________________________

Name of left neighbor: _____________________________
Name of right neighbor: _____________________________

Instructions:

Show your work. An answer without explanation is not acceptable and does not guarantee any credit.

Only the front pages will be scanned and graded. Back pages won't be scanned; you can use them as scratch paper.

Do not remove pages, as this disrupts the scanning. If needed, cross out any parts that you don't want us to grade.

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>MAX</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
</tbody>
</table>
Problem 1 Warm up (15 points)

a) Consider the following circuit.

\[ d^2v_{\text{out}} \left( \frac{dt}{dt} \right) + A \frac{dv_{\text{out}}}{dt} + Bv_{\text{out}} = 0 \]

If \( A < 2\sqrt{B} \), provide an expression for \( v_{\text{out}}(t) \geq 0 \). (5 points)

\[ \alpha = \frac{A}{2}, \quad \omega_0 = \sqrt{B} \quad \text{thus} \quad \alpha < \omega_0 \]

The circuit is underdamped.

\[ v(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \]

\[ \lambda_1, \lambda_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

\[ v(0) = k_1 + k_2 = v_0 \quad \text{Two eqns., two unknowns} \]

\[ \frac{dv}{dt}(0) = \lambda_1 k_1 + \lambda_2 k_2 = 0 \]

Solution:

\[ v_{\text{out}}(t) = \frac{\lambda_1}{\lambda_1 - \lambda_2} \]

\[ k_1 = \frac{\lambda_2}{\lambda_1 - \lambda_2} v_0 \]

or

\[ v_{\text{out}}(t) = e^{-\frac{A}{2}t} \left[ (K_1 + K_2) \cos \left( \sqrt{B - \frac{A^2}{4}} t \right) + j (K_1 - K_2) \sin \left( \sqrt{B - \frac{A^2}{4}} t \right) \right] \]
b) Consider the circuit below.

\[ \tilde{H}_{\text{out}}(\omega) = \frac{\tilde{v}_{\text{out}}(\omega)}{\tilde{v}_{\text{in}}(\omega)} \text{ for } \omega \to \infty? \] (5 points)

Solution:

\[ \tilde{H}_{\text{out}}(\omega \to \infty) = 0 \]
c) Consider the Bode plot below. (5 points)

This is a Bode magnitude plot of the transfer function $\tilde{H}(\omega)$. The expression for $\tilde{H}(\omega)$ is shown below.

$$
\tilde{H}(\omega) = \frac{\tilde{H}_x(\omega)}{1 + j\left(\frac{\omega}{\omega_c}\right)}
$$

What is $\tilde{H}_x(\omega)$?

Solution:

$$
\tilde{H}_x(\omega) = \pm 10
$$

$$
20 \log_{10} (|H|) = 20 \\
10 \log_{10} (|H|) = 1 \\
|H| = 10
$$
Problem 2 H's and Bodes... (25 points)

Consider the circuit below. There is nothing connected to the $V_{out}$ terminal. (5 points)

![Circuit Diagram]

a) Provide an expression for $\tilde{H}_{out}(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)}$

\[
\tilde{H}(\omega) = \frac{Z_3}{Z_1 + Z_3}
\]

Solution:

$\tilde{H}_{out}(\omega) = \frac{Z_3}{Z_1 + Z_3}$
b) For this part of the problem, assume you have ONE capacitor, ONE inductor and ONE resistor. If \( Z_2 = 0 \) for all \( \omega \), which components would you choose for \( Z_1 \) and \( Z_3 \) such that the filter response is a passive low pass filter with a slope of -20 dB/decade for frequencies beyond a single cutoff frequency? (5 points)

**Solution:**

Either combo is correct

Circle ONE component to go into the \( Z_1 \) box: Capacitor

Circle ONE component to go into the \( Z_3 \) box: Capacitor

\[
H(w) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega \frac{1}{RC}}
\]

\( \omega_c = \frac{1}{RC} \)

\[
H(w) = \frac{R}{R + j\omega L} = \frac{R}{\frac{R}{R} \frac{1}{1 + j\omega \frac{R}{L}}} = \frac{1}{1 + j\omega \frac{R}{L}}
\]

\( \omega_c = \frac{R}{L} \)
c) Consider the following circuit:

We define a transfer function \( \hat{H}_{out}(\omega) = \frac{I_{out}(\omega)}{V_{in}(\omega)} \).

**LOOK AT THE DEFINITION OF THE TRANSFER FUNCTION CAREFULLY.**

Provide an expression in *canonical form* for \( \hat{H}_{out}(\omega) \). (10 points)

\[
\frac{\hat{V}_x}{\hat{V}_{in}} = \frac{jwL \parallel R_2}{R_1 + jwL \parallel R_2} = \frac{jwLR_2}{jwLR_2 + R_1(R_2 + jwL)}
\]

\[
\frac{\hat{I}_{out}}{\hat{V}_x} = \frac{1}{R_2}
\]

\[
\frac{jwL \parallel R_2}{R_2 + jwL} = \frac{jwLR_2}{R_2(R_2 + jwL)}
\]

Solution:

\[
\hat{H}_{out}(\omega) = \frac{jwL}{R_1R_2 + jwL(R_1+R_2)}
\]

\[
K = \frac{1}{R_1R_2}
\]

\[
\omega c_1 = \frac{1}{L}
\]

\[
\omega c_2 = \frac{R_1R_2}{L(R_1+R_2)}
\]

\[
\frac{1}{1 + jw/\omega c_2}
\]
d) If $L = 1 \, \text{H}$ and $R_1 = R_2 = 1 \, \Omega$, provide below magnitude and phase Bode plots for $\tilde{H}_{\text{out}}(\omega)$. (5 points)
Problem 3 Transistors and RC’s (15 points)
Consider the circuit below.

\[ \Rightarrow V_{out} = \overline{A} \overline{B} \]

a) Fill in the truth table below for the circuit above. \( V_A \), \( V_B \) and \( V_{out} \) are digital voltages that can only assume values of 0 or \( V_{DD} \) (5 points)

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_A )</td>
<td>( V_B )</td>
<td>( V_{out} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>( V_{DD} )</td>
</tr>
<tr>
<td>0</td>
<td>( V_{DD} )</td>
<td>( V_{DD} )</td>
</tr>
<tr>
<td>( V_{DD} )</td>
<td>0</td>
<td>( V_{DD} )</td>
</tr>
<tr>
<td>( V_{DD} )</td>
<td>( V_{DD} )</td>
<td>0</td>
</tr>
</tbody>
</table>
For this part, assume that $V_A = V_B = V_{DD}$ for $-\infty < t < 0$. At $t=0$, $V_A$ and $V_B$ switch instantly from $V_{DD}$ to 0. Assume all transistors behave as resistors with the same value, $R$, if in the ON state and that all capacitances are already accounted for in the circuit above.

b) Provide an expression for $V_{out}(t \geq 0)$. (10 points)

$$V_{DD} - V_{out} = C_L \frac{dV_c}{dt}$$

$$V_c = V_{out}$$

$$\frac{2}{RCL} (V_{DD} - V_{out}) = \frac{dV_{out}}{dt} \Rightarrow \frac{2}{RCL} \frac{dV_{out}}{dt} = \frac{V_{DD} - V_{out}}{t}$$

$$V_{out} = V_{DD} - V_{out} \Rightarrow V_{out} = V_{DD} - \tilde{V}_{out}$$

$$\tilde{V}_{out} = \frac{-2e^{-t/RC}}{-2\tilde{V}_{out}} + V_{DD}$$

$$V_{out}(0) = 0 = V_{DD} - C \Rightarrow C = V_{DD}$$

Solution:

$$V_{out}(t) = V_{DD} (1 - e^{-2t/RC})$$
Problem 4 Phasors! (20 points)
Consider the circuit below.

We are going to solve this circuit, which contains both a sinusoidal and a DC source using superposition and phasors.

a) Solve for $v_{out}(t)$ if $i_{in}(t) = 0$ and $V_{DD}$ is a non-zero constant. (5 points)

$$V_{DD} \ \text{const} \Rightarrow \text{steady state analysis} \Rightarrow L=\text{short} \ \ C=\text{open}$$

$$i_{in}=0 \Rightarrow \text{acts as an open circuit}$$

$$V_{out}=\frac{R_{out}}{R_{p}+R_{out}}V_{DD}$$

Solution:

$$v_{out}(t)=\frac{R_{out}}{R_{p}+R_{out}}V_{DD}$$
b) Solve for $v_{out}(t)$ if $i_{in}(t) = I_0 \cos(\omega t)$ and $V_{DD} = 0 \ V$. (10 points)

\[ v_{in}(t) = I_0 e^{j\omega t} \]
\[ Z_L = j\omega L \]
\[ Z_C = \frac{1}{j\omega C} \]

**Current divider:**
\[ \tilde{i}_0 = \frac{R_P \parallel Z_C}{R_P \parallel Z_C + Z_L + R_{out}} \tilde{i}_{in} \]

\[ \tilde{v}_{out} = \tilde{i}_0 \cdot R_{out} = \frac{R_P \parallel Z_C}{R_P \parallel Z_C + Z_L + R_{out}} \cdot R_{out} \cdot \tilde{i}_{in} \]

\[ \tilde{v}_{out} = \frac{R_P \cdot R_{out} + j\omega C R_P}{1 + j\omega C R_P} \]

\[ \tilde{v}_{out} = \frac{\sqrt{(R_P + R_{out})^2 + (\omega L + \omega C R_P)^2}}{R_P + R_{out} + j\omega (L + C R_P) + j\omega^2 C R_P} \]

\[ v_{out}(t) = R_e \left( \tilde{v}_{out} e^{j\omega t} \right) = |\tilde{v}_{out}| \cos(\omega t + \angle \tilde{v}_{out}) \]

**Solution:**
\[ v_{out}(t) = \frac{I_0 R_{out} + R_P}{\sqrt{(R_P + R_{out})^2 + (\omega L + \omega C R_P)^2}} \]

\[ \cos(\omega t - \tan^{-1}\left(\frac{\omega L + \omega C R_P}{R_P + R_{out} - \omega^2 C R_P}\right)) \]
c) Solve for $V_{OUT}(t)$ if $i_{IN}(t) = I_0 \cos(\omega t)$ and $V_{DD}$ = a non-zero constant. (5 points)

Superposition, sol. from part a + sol. from part b

Solution:

$$V_{OUT}(t) = V_{OUT}(t)_{\text{from (a)}} + V_{OUT}(t)_{\text{from (b)}}$$
Problem 5 (25 points)
a) Consider the following circuit. The switch is closed for t<0, then opens at t=0. Both of the independent sources have a DC value (i.e. they do not change with time).

![Circuit Diagram]

a) What is $i_x(t<0)$? (5 points)

Steady state $\Rightarrow$ L shorts, C opens
L shorts out $R_x$, $C_1R$

**Solution:**

$$I_1$$

b) What is $v_y(t<0)$? (5 points)

Same ckt as (a)

**Solution:**

$$v_y = R_x \cdot I_1$$
c) Provide an equation in the variable $i_x(t)$ that, when solved, would provide an expression for $i_x(t)$ for $t \geq 0$. 
DO NOT SOLVE THE EQUATION. (10 points)

\[
\begin{align*}
\text{Solution:} & \\
ix + \frac{1}{R} \frac{dix}{dt} + \frac{1}{L} \frac{d^2 ix}{dt^2} &= 0
\end{align*}
\]

\[
\begin{align*}
d) \text{ If } Cx = Cy = C = 1 \text{ F and } R_x = R = 1 \text{ } \Omega \text{ and } L_y = L = 1 \text{ H, provide an expression for } i_x(t) \text{ for } t \geq 0. (5 \text{ points})
\end{align*}
\]

\[
\begin{align*}
ix + \frac{dix}{dt} + \frac{d^2 ix}{dt^2} &= 0
\end{align*}
\]

\[
\begin{align*}
\omega &= \frac{1}{\sqrt{LC}} \\
\omega_0 &= 1 \\
\chi &= \frac{-1}{2} \pm \sqrt{\frac{1}{2} - 1} \\
ix(t) &= C_1 e^{\chi_1 t} + C_2 e^{\chi_2 t}
\end{align*}
\]

\[
\begin{align*}
i_x(0) &= I_1 = C_1 + C_2 \\
\frac{dix}{dt}(0) &= \chi_1 C_1 + \chi_2 C_2
\end{align*}
\]

\[
\begin{align*}
\text{Solution:} & \\
i_x(t) &= C_1 e^{\chi_1 t} + C_2 e^{\chi_2 t}
\end{align*}
\]

\[
\begin{align*}
\text{OR} \\
i_x(t) &= e^{\frac{-t}{2}} \left[ (C_1 + C_2) \cos\left(\sqrt{\frac{3}{4}} \ t\right) + j(C_1 - C_2) \sin\left(\sqrt{\frac{3}{4}} \ t\right) \right]
\end{align*}
\]
<table>
<thead>
<tr>
<th>Factor</th>
<th>Bode Magnitude</th>
<th>Bode Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant K</td>
<td>$20 \log K$</td>
<td>$\pm 180^\circ$ if $K &lt; 0$</td>
</tr>
<tr>
<td></td>
<td>$0 \text{ dB}$</td>
<td>$0^\circ$ if $K &gt; 0$</td>
</tr>
<tr>
<td>Zero @ Origin</td>
<td>$(j\omega)^N$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0 \text{ dB}$</td>
<td>$(90N)^\circ$</td>
</tr>
<tr>
<td>Pole @ Origin</td>
<td>$(j\omega)^{-N}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0 \text{ dB}$</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(-90N)^\circ$</td>
</tr>
<tr>
<td>Simple Zero</td>
<td>$(1 + j\omega/\omega_c)^N$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0 \text{ dB}$</td>
<td>$(90N)^\circ$</td>
</tr>
<tr>
<td>Simple Pole</td>
<td>$\left( \frac{1}{1 + j\omega/\omega_c} \right)^N$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0 \text{ dB}$</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>Quadratic Zero</td>
<td>$[1 + j2\xi\omega_c + (j\omega/\omega_c)^2]^N$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0 \text{ dB}$</td>
<td>$(180N)^\circ$</td>
</tr>
<tr>
<td>Quadratic Pole</td>
<td>$\frac{1}{[1 + j2\xi\omega_c + (j\omega/\omega_c)^2]^N}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0 \text{ dB}$</td>
<td>$(-180N)^\circ$</td>
</tr>
</tbody>
</table>
Contributors:

- Murat Arcak.
- Justin Yim.
- Kyoungtae Lee.
- Nikhil Shinde.