This homework is due on Thursday, March 15, 2018, at 11:59AM (NOON). Self-grades are due on Monday, March 19, 2018, at 11:59AM (NOON).

Pre-Lab

1. Open-Loop Control of SIXT33N

Last time, we learned that the ideal input PWM for running a motor at a target velocity $v^*$ is:

$$u(t) = \frac{v^* + \beta}{\theta}$$

In this problem, we will extend our analysis from one motor to a two-motor car system and evaluate how well our open-loop control scheme does.

$$v_L(t) = d_L(t+1) - d_L(t) = \theta_L u_L(t) - \beta_L$$
$$v_R(t) = d_R(t+1) - d_R(t) = \theta_R u_R(t) - \beta_R$$

(a) In reality, we need to “kickstart” electric motors with a pulse in order for them to work. That is, we can’t go straight from 0 to our desired input signal for $u(t)$, since the motor needs to overcome its initial inertia in order to operate in accordance with our model.

Let us model the pulse as having a width (in timesteps) of $t_p$. In order to model this phenomenon, we can say that $u(t) = 255$ for $t \in [0, t_p - 1]$. In addition, the car initially (at $t = 0$) hasn’t moved, so we can also say $d(0) = 0$.

Firstly, let us examine what happens to $d_L$ and $d_R$ at $t = t_p$, that is, after the kickstart pulse has passed. Find $d_L(t_p)$ and $d_R(t_p)$. (Hint: If it helps, try finding $d_L(1)$ and $d_R(1)$ first and then generalizing your result to the $t_p$ case.)

Note: It is very important that you distinguish $\theta_L$ and $\theta_R$ as the motors we have are liable to vary in their parameters, just as how real resistors vary from their ideal resistance.

(b) Let us define $\delta(t) = d_L(t) - d_R(t)$ as the difference in positions between the two wheels. If both wheels of the car are going at the same velocity, then this difference $\delta$ should remain constant since no wheel will advance by more ticks than the other. As a result, this will be useful in our analysis and in designing our control schemes.

Find $\delta(t_p)$. For both an ideal car ($\theta_L = \theta_R$ and $\beta_L = \beta_R$) where both motors are perfectly ideal and a non-ideal car ($\theta_L \neq \theta_R$ and $\beta_L \neq \beta_R$), did the car turn compared to before the pulse?

Note: Since $d(0) = d_L(0) = d_R(0) = 0$, $\delta(0) = 0$. 

\(^1x \in [a, b]\) means that $x$ goes from $a$ to $b$ inclusive.
(c) We can still declare victory though, even if the car turns a little bit during the initial pulse ($t_p$ will be very short in lab), so long as the car continues to go straight afterwards when we apply our control scheme; that is, as long as $\delta(t \to \infty)$ converges to a constant value (as opposed to going to $\pm\infty$ or oscillating).

Let’s try applying the open-loop control scheme we learned last week to each of the motors independently, and see if our car still goes straight.

$$u_L(t) = \frac{v^* + \beta_L}{\theta_L}$$
$$u_R(t) = \frac{v^* + \beta_R}{\theta_R}$$

Let $\delta(t_p) = \delta_0$. Find $\delta(t)$ for $t \geq t_p$ in terms of $\delta_0$. (Hint: As in part (a), if it helps you, try finding $\delta(t_p + 1)$, $\delta(t_p + 2)$, etc., and generalizing your result to the $\delta(t)$ case.)

Does $\delta(t \to \infty)$ deviate from $\delta_0$? Why or why not?

(d) Unfortunately, in real life, it is hard to capture the precise parameters of the car motors like $\theta$ and $\beta$, and even if we did manage to capture them, they could vary as a function of temperature, time, wheel conditions, battery voltage, etc. In order to model this effect of model mismatch, we consider model mismatch terms (such as $\Delta \theta_L$), which reflects the discrepancy between the model parameters and actual parameters.

$$v_L(t) = d_L(t+1) - d_L(t) = (\theta_L + \Delta \theta_L)u_L(t) - (\beta_L + \Delta \beta_L)$$
$$v_R(t) = d_R(t+1) - d_R(t) = (\theta_R + \Delta \theta_R)u_R(t) - (\beta_R + \Delta \beta_R)$$

Let us try applying the open-loop control scheme again to this new system. Note that no model mismatch terms appear below – this is intentional since our control scheme is derived from the model parameters for $\theta$ and $\beta$, not from the actual $\theta + \Delta \theta$, etc.

$$u_L(t) = \frac{v^* + \beta_L}{\theta_L}$$
$$u_R(t) = \frac{v^* + \beta_R}{\theta_R}$$

As before, let $\delta(t_p) = \delta_0$. Find $\delta(t)$ for $t \geq t_p$ in terms of $\delta_0$.

Does $\delta(t \to \infty)$ change from $\delta_0$? Why or why not, and how is it different from the previous case of no model mismatch?

You may have noticed that open-loop control is insufficient in light of non-idealities and mismatches. Next time, we will analyze a more powerful form of control (closed-loop control) which should be more robust against these kinds of problems.

Problems

2. Controllability and Discretization

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2Why not just do a better job of capturing the parameters, one may ask? Well, as noted above, the mismatch can vary as a function of an assortment of factors including temperature, time, wheel conditions, battery voltage, and it is not realistic to try to capture the parameters under every possible environment, so it is up to the control designer to ensure that the system can tolerate a reasonable amount of mismatch.
In this problem, we will use the car model

\[
\frac{d}{dt} p(t) = v(t) \\
\frac{d}{dt} v(t) = u(t)
\]

that was discussed in class.

(a) Assuming that the input \( u(t) \) can be varied continuously, is this system controllable?

(b) Now assume that we can only change our control input every \( T \) seconds. Derive a discrete-time state space model for the state updates, assuming that the input is held constant between times \( t \) and \( t + T \).

(c) Is the discrete-time system controllable?

3. Controllability in Circuits

Consider the circuit in Figure 1 where \( V_s \) is an input we can control:

![Figure 1: Controllability in circuits](image)

(a) Write out the state space model for this circuit using \( V_1 \) and \( V_2 \) as the state variables.

(b) Show that this system is not controllable.

(c) Explain, in terms of circuit currents and voltages why this system isn’t controllable. (Hint: Think about what currents/voltages of the circuit we are controlling with \( V_s \).)

(d) Draw an equivalent circuit of this system that is controllable. What quantity can you control in this system?

4. Controllability in 2D

Consider the control of some two-dimensional linear discrete-time system

\[
\bar{x}[k+1] = A\bar{x}[k] + Bu[k]
\]

where \( A \) is a \( 2 \times 2 \) real matrix and \( B \) is a \( 2 \times 1 \) real vector.

(a) Let \( A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \) with \( a, c, d \neq 0 \), and \( B = \begin{bmatrix} f \\ g \end{bmatrix} \). Find a \( B \) such that the system is controllable no matter what nonzero values \( a, c, d \) take on, and a \( B \) for which it is not controllable no matter what nonzero values are given for \( a, c, d \). You may use the controllability rank test, but explain your intuition as well.
(b) Let \( A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \) with \( a, d \neq 0 \). and \( B = \begin{bmatrix} f \\ g \end{bmatrix} \) with \( f, g \neq 0 \). Is this system always controllable? If not, find configurations of nonzero \( a, d, f, g \) that make the system uncontrollable.

(c) We want to see if controllability is preserved under changes of coordinates. To begin with, let \( \tilde{z}[k] = V^{-1} \tilde{x}[k] \), write out the system equation with respect to \( \tilde{z} \).

(d) Now show that controllability is preserved under change of coordinates. (Hint: Use the fact that \( \text{rank}(MA) = \text{rank}(A) \) for any invertible matrix \( M \).)

5. Buoyancy

An engineer would like to deploy an autonomous communications balloon (like Project Loon’s balloons: https://plus.google.com/+ProjectLoon/posts/PVitgyeYweY) to provide internet connectivity to a particular geographical region. To provide reliable connectivity, the balloon must hold its position over the region it services. The balloon can control its altitude \( (a) \) by changing its buoyancy, but it doesn’t have any engines. In order to move horizontally (horizontal position \( p \)), the balloon drifts on air currents.

Consulting meteorologists, the engineer has modeled the air currents around the desired balloon position (the point \((0,0)\)) and found the flow field shown in Figure 2.

![Image of Project Loon Balloon](image1)

![Image of Wind Speeds](image2)

**Figure 2: Project Loon Balloon**

**Figure 3: Wind speeds**

where the wind speed at each point is described by the equations:

\[
\begin{align*}
    v_p &= -20p + 20a \\
    v_a &= -20p + 20a
\end{align*}
\]

where the velocities are in kilometers per hour and the horizontal position and altitude are in kilometers. Putting this together with the balloon’s buoyancy control, the balloon’s dynamics are described by:

\[
\begin{bmatrix}
\dot{p} \\
\dot{a}
\end{bmatrix} = \begin{bmatrix} -20 & 20 \\ -20 & 20 \end{bmatrix} \begin{bmatrix} p \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u
\]
(a) Write the dynamics equation in controller canonical form.

(b) What is the matrix $T$ for the change of variables $\vec{z} = T\vec{x}$ that transforms the original $A$ and $B$ matrices into controller canonical form?

(c) The engineer would like the balloon to converge to $(0,0)$ with eigenvalues $-1$ and $-1$. What should be the state feedback gains $K$ multiplying the original state vector $\vec{x}$ to achieve this behavior? Write out the expression for $u$ in terms of $p$ and $a$.

6. CCR Circuit

Consider the circuit below driven by a current source with current $u(t)$. The output $y(t)$ is the voltage across the resistor and the state variables are the capacitor voltages as marked in the circuit diagram.

![Figure 4: Two Capacitor Circuit with Current Source](image)

(a) Write a state model for this circuit.

(b) Find all equilibrium points when $u(t) = 0$ for all $t$.

(c) Determine if the system is controllable.

(d) OPTIONAL: Determine if the system is observable.

(e) OPTIONAL: If your answer to part (c) or (d) is no, explain the physical reason for lack of controllability or observability, whichever is applicable.

7. Inverted Pendulum on a Rolling Cart

Consider the inverted pendulum depicted below, which is placed on a rolling cart and whose equations of motion are given by:

$$\dddot{y} = \frac{1}{M + m + \sin^2 \theta} \left( \frac{u}{m} + \dot{\theta}^2 \ell \sin \theta - g \sin \theta \cos \theta \right)$$

$$\ddot{\theta} = \frac{1}{\ell (M + m + \sin^2 \theta)} \left( -\frac{u}{m} \cos \theta - \dot{\theta}^2 \ell \cos \theta \sin \theta + \frac{M + m}{m} g \sin \theta \right).$$
Consider the inverted pendulum system depicted below.

(a) Write the state model using the variables \( x_1(t) = \theta(t) \), \( x_2(t) = \dot{\theta}(t) \), and \( x_3(t) = \dot{y}(t) \). We do not include \( y(t) \) as a state variable because we are interested in stabilizing at the point \( \theta = 0 \), \( \dot{\theta} = 0 \), \( \dot{y} = 0 \), and we are not concerned about the final value of the position \( y(t) \).

(b) Linearize this model at the equilibrium \( x_1 = 0 \), \( x_2 = 0 \), \( x_3 = 0 \), and \( u = 0 \), and indicate the resulting \( A \) and \( B \) matrices.

(c) Show that the linearized model is controllable.

(d) Suppose \( M = 1 \), \( m = 0.1 \), \( l = 1 \), and \( g = 10 \), and design a state feedback controller,

\[
  u(t) = k_1 \theta(t) + k_2 \dot{\theta}(t) + k_3 \dot{y}(t),
\]

such that the eigenvalues of \( A - BK \) (the “closed-loop eigenvalues”) are \( \lambda_1 = \lambda_2 = \lambda_3 = -1 \).

(e) Suppose we set \( k_2 = k_3 = 0 \) and vary only \( k_1 \); that is, the controller uses only \( \theta(t) \) for feedback. Does there exist a \( k_1 \) value such that all closed-loop eigenvalues have negative real parts?

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