This homework is due on Thursday, March 15, 2018, at 11:59AM (NOON). Self-grades are due on Monday, March 19, 2018, at 11:59AM (NOON).

Pre-Lab

1. Open-Loop Control of SIXT33N

   Last time, we learned that the ideal input PWM for running a motor at a target velocity \( v^* \) is:

   \[
   u(t) = \frac{v^* + \beta}{\theta}
   \]

   In this problem, we will extend our analysis from one motor to a two-motor car system and evaluate how well our open-loop control scheme does.

   \[
   v_L(t) = d_L(t + 1) - d_L(t) = \theta_L u_L(t) - \beta_L
   \]

   \[
   v_R(t) = d_R(t + 1) - d_R(t) = \theta_R u_R(t) - \beta_R
   \]

   (a) In reality, we need to “kickstart” electric motors with a pulse in order for them to work. That is, we can’t go straight from 0 to our desired input signal for \( u(t) \), since the motor needs to overcome its initial inertia in order to operate in accordance with our model.

   Let us model the pulse as having a width (in timesteps) of \( t_p \). In order to model this phenomenon, we can say that \( u(t) = 255 \) for \( t \in [0, t_p - 1] \). In addition, the car initially (at \( t = 0 \)) hasn’t moved, so we can also say \( d(0) = 0 \).

   Firstly, let us examine what happens to \( d_L \) and \( d_R \) at \( t = t_p \), that is, after the kickstart pulse has passed. Find \( d_L(t_p) \) and \( d_R(t_p) \). (Hint: If it helps, try finding \( d_L(1) \) and \( d_R(1) \) first and then generalizing your result to the \( t_p \) case.)

   Note: It is very important that you distinguish \( \theta_L \) and \( \theta_R \) as the motors we have are liable to vary in their parameters, just as how real resistors vary from their ideal resistance.

   Solution:

   Applying the model directly, we get:

   \[
   d(1) = d(0) + (255\theta - \beta)
   \]

   \[
   d(2) = d(1) + (255\theta - \beta) = d(0) + (255\theta - \beta) + (255\theta - \beta) = d(0) + 2(255\theta - \beta)
   \]

   \[
   d(3) = d(2) + (255\theta - \beta) = d(0) + 2(255\theta - \beta) + (255\theta - \beta) = d(0) + 3(255\theta - \beta)
   \]

   \[
   d(t_p) = d(0) + t_p(255\theta - \beta) \quad \text{(by analogy)}
   \]

   \[
   d(t_p) = t_p(255\theta - \beta) \quad \text{(substitute \( d(0) \))}
   \]

---

1. \( x \in [a, b] \) means that \( x \) goes from \( a \) to \( b \) inclusive.
Thus we get:

\[ d_L(t_p) = t_p(255\theta_L - \beta_L) \]
\[ d_R(t_p) = t_p(255\theta_R - \beta_R) \]

(b) Let us define \( \delta(t) = d_L(t) - d_R(t) \) as the difference in positions between the two wheels. If both wheels of the car are going at the same velocity, then this difference \( \delta \) should remain constant since no wheel will advance by more ticks than the other. As a result, this will be useful in our analysis and in designing our control schemes.

Find \( \delta(t_p) \). For both an ideal car (\( \theta_L = \theta_R \) and \( \beta_L = \beta_R \)) where both motors are perfectly ideal and a non-ideal car (\( \theta_L \neq \theta_R \) and \( \beta_L \neq \beta_R \)), did the car turn compared to before the pulse?

**Note:** Since \( d(0) = d_L(0) = d_R(0) = 0 \), \( \delta(0) = 0 \).

**Solution:**

\[ \delta(t_p) = d_L(t_p) - d_R(t_p) \]
\[ \delta(t_p) = t_p(255\theta_L - \beta_L) - t_p(255\theta_R - \beta_R) \]
\[ \delta(t_p) = t_p(255\theta_L - \beta_L - (255\theta_R - \beta_R)) \]
\[ \delta(t_p) = t_p(255(\theta_L - \theta_R) - (\beta_L - \beta_R)) \]

For an ideal car, both the \( \theta_L - \theta_R \) and \( \beta_L - \beta_R \) terms go to zero, so the pulse made the car go perfectly straight. However, in the non-ideal car, we aren’t so lucky, since the car did turn somewhat during the initial pulse.

(c) We can still declare victory though, even if the car turns a little bit during the initial pulse (\( t_p \) will be very short in lab), so long as the car continues to go straight afterwards when we apply our control scheme; that is, as long as \( \delta(t \rightarrow \infty) \) converges to a constant value (as opposed to going to \( \pm \infty \) or oscillating).

Let’s try applying the open-loop control scheme we learned last week to each of the motors independently, and see if our car still goes straight.

\[ u_L(t) = \frac{v^* + \beta_L}{\theta_L} \]
\[ u_R(t) = \frac{v^* + \beta_R}{\theta_R} \]

Let \( \delta(t_p) = \delta_0 \). Find \( \delta(t) \) for \( t \geq t_p \) in terms of \( \delta_0 \). (**Hint:** As in part (a), if it helps you, try finding \( \delta(t_p + 1), \delta(t_p + 2) \), etc., and generalizing your result to the \( \delta(t) \) case.)

Does \( \delta(t \rightarrow \infty) \) deviate from \( \delta_0 \)? Why or why not?

**Solution:**
\[ \delta(t_p + 1) = d_L(t_p + 1) - d_R(t_p + 1) \\
= d_L(t_p) + \theta_L u(t) - \beta_L - (d_R(t_p) + \theta_R u(t) - \beta_R) \\
= d_L(t_p) + \theta_L u(t) - \beta_L - d_R(t_p) - \theta_R u(t) + \beta_R \\
= (d_L(t_p) - d_R(t_p)) + (\theta_L u(t) - \beta_L) - (\theta_R u(t) - \beta_R) \\
= (d_L(t_p) - d_R(t_p)) + v^* - v^* \\
= (d_L(t_p) - d_R(t_p)) \\
= \delta(t) = \delta_0 \] (by generalization: every step does not change \( \delta \))

Since we are able to apply just the right amount of input PWM to keep a constant velocity on both wheels, neither wheel gets ahead of the other, so \( \delta(t) \) does not change, meaning that the car does not turn.

(d) Unfortunately, in real life, it is hard to capture the precise parameters of the car motors like \( \theta \) and \( \beta \), and even if we did manage to capture them, they could vary as a function of temperature, time, wheel conditions, battery voltage, etc. In order to model this effect of model mismatch, we consider model mismatch terms (such as \( \Delta \theta_L \)), which reflects the discrepancy between the model parameters and actual parameters.

\[ v_L(t) = d_L(t + 1) - d_L(t) = (\theta_L + \Delta \theta_L) u_L(t) - (\beta_L + \beta_L) \]
\[ v_R(t) = d_R(t + 1) - d_R(t) = (\theta_R + \Delta \theta_R) u_R(t) - (\beta_R + \beta_R) \]

Let us try applying the open-loop control scheme again to this new system. Note that no model mismatch terms appear below – this is intentional since our control scheme is derived from the model parameters for \( \theta \) and \( \beta \), not from the actual \( \theta + \Delta \theta \), etc.

\[ u_L(t) = \frac{v^* + \beta_L}{\theta_L} \]
\[ u_R(t) = \frac{v^* + \beta_R}{\theta_R} \]

As before, let \( \delta(t_p) = \delta_0 \). Find \( \delta(t) \) for \( t \geq t_p \) in terms of \( \delta_0 \).

Does \( \delta(t \to \infty) \) change from \( \delta_0 \)? Why or why not, and how is it different from the previous case of no model mismatch?

**Solution:**

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2Why not just do a better job of capturing the parameters, one may ask? Well, as noted above, the mismatch can vary as a function of an assortment of factors including temperature, time, wheel conditions, battery voltage, and it is not realistic to try to capture the parameters under every possible environment, so it is up to the control designer to ensure that the system can tolerate a reasonable amount of mismatch.
\[
\delta(t_{p} + 1) = d_L(t_{p} + 1) - d_R(t_{p} + 1)
\]
\[
= (d_L(t_{p}) + (\theta_L + \Delta \theta_L)u_L(t) - (\beta_L + \Delta \beta_L)) - (d_R(t_{p}) + (\theta_R + \Delta \theta_R)u_R(t) - (\beta_R + \Delta \beta_R))
\]
\[
= (d_L(t_{p}) - d_R(t_{p})) + ((\theta_L + \Delta \theta_L)u_L(t) - (\beta_L + \Delta \beta_L)) - ((\theta_R + \Delta \theta_R)u_R(t) - (\beta_R + \Delta \beta_R))
\]
\[
= \delta(t_{p}) + ((\theta_L + \Delta \theta_L)u_L(t) - (\beta_L + \Delta \beta_L)) - ((\theta_R + \Delta \theta_R)u_R(t) - (\beta_R + \Delta \beta_R))
\]
\[
= \delta_0 + (\theta_L u_L(t) - \beta_L + \Delta \theta_L u_L(t) - \Delta \beta_L) - (\theta_R u_R(t) - \beta_R + \Delta \theta_R u_R(t) - \Delta \beta_R)
\]
\[
= \delta_0 + (v^* + \Delta \theta_L u_L(t) - \Delta \beta_L) - (v^* + \Delta \theta_R u_R(t) - \Delta \beta_R)
\]
\[
= \delta_0 + v^* - v^* + (\Delta \theta_L u_L(t) - \Delta \beta_L) - (\Delta \theta_R u_R(t) - \Delta \beta_R)
\]
\[
= \delta_0 + (\Delta \theta_L u_L(t) - \Delta \beta_L) - (\Delta \theta_R u_R(t) - \Delta \beta_R)
\]
\[
= \delta_0 + \left( \frac{\Delta \theta_L}{\theta_L} (v^* + \beta_L) - \Delta \beta_L \right) - \left( \frac{\Delta \theta_R}{\theta_R} (v^* + \beta_R) - \Delta \beta_R \right)
\]
\[
\delta(t) = \delta_0 + (t - t_{p}) \left( \frac{\Delta \theta_L}{\theta_L} (v^* + \beta_L) - \Delta \beta_L \right) - \left( \frac{\Delta \theta_R}{\theta_R} (v^* + \beta_R) - \Delta \beta_R \right)
\]

If there is no model mismatch (i.e., all mismatch terms are zero), then we are back to the same case as last time (all those terms drop out, and \( \delta \) does not change).

If there is model mismatch, however, we are not so lucky.

As \( t \to \infty \), the term \( \left( \frac{\Delta \theta_L}{\theta_L} (v^* + \beta_L) - \Delta \beta_L \right) - \left( \frac{\Delta \theta_R}{\theta_R} (v^* + \beta_R) - \Delta \beta_R \right) \) (which is highly unlikely to be zero) causes \( \delta \) to either steadily increase or decrease, meaning that the car turns steadily more and more.

You may have noticed that open-loop control is insufficient in light of non-idealities and mismatches. Next time, we will analyze a more powerful form of control (closed-loop control) which should be more robust against these kinds of problems.

**Problems**

2. **Controllability and Discretization**

   In this problem, we will use the car model

   \[
   \frac{dp}{dt} = v(t)
   \]

   \[
   \frac{dv}{dt} = u(t)
   \]

   that was discussed in class.

   (a) Assuming that the input \( u(t) \) can be varied continuously, is this system controllable?

   **Solution:**

   Introducing states \( x_1 = p \) and \( x_2 = v \), we rewrite this system in state space form

   \[
   \dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).
   \]
The controllability matrix
\[
\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
has rank 2. Therefore, the continuous-time system is controllable.

(b) Now assume that we can only change our control input every \( T \) seconds. Derive a discrete-time state space model for the state updates, assuming that the input is held constant between times \( t \) and \( t + T \).

**Solution:**
By integrating both sides of the second equation from \( t \) to \( t + T \) and keeping in mind that \( u(t) \) is constant in this interval, that is \( u(t + \tau) = u(t) \) for \( \tau \in [0,T) \):
\[
v(t + T) - v(t) = \int_{0}^{T} u(t + \tau) d\tau = Tu(t).
\]
Now integrating the first equation and using the fact that \( v(t + \tau) = v(t) + \tau u(t) \) we get
\[
p(t + T) - p(t) = \int_{0}^{T} (v(t) + \tau u(t)) d\tau = Tv(t) + \frac{1}{2}T^2 u(t).
\]
Introducing states \( x_1[k] = p(kT) \) and \( x_2[k] = v(kT) \) we get the state space model
\[
\dot{x}[k+1] = A\dot{x}[k] + Bu[k] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \dot{x}[k] + \begin{bmatrix} 1 \frac{T^2}{2} \\ T \end{bmatrix} u[k].
\]

(c) Is the discrete-time system controllable?

**Solution:**
The controllability matrix
\[
\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{1}{2}T^2 & \frac{3}{2}T^2 \\ T & T \end{bmatrix}
\]
is full rank, so the discrete-time system is controllable.

3. Controllability in Circuits
Consider the circuit in Figure 1 where \( V_s \) is an input we can control:

![Controllability in circuits](image)

**Figure 1**: Controllability in circuits
(a) Write out the state space model for this circuit using $V_1$ and $V_2$ as the state variables.

**Solution:**

\[
I = \frac{V_s - V_1 - V_2}{R} = C_1 \frac{dV_1}{dt} = C_2 \frac{dV_2}{dt}
\]

\[
\frac{d}{dt} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC_1} & -\frac{1}{RC_2} \\ -\frac{1}{RC_2} & -\frac{1}{RC_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{RC_2} \end{bmatrix} V_s
\]

(b) Show that this system is not controllable.

**Solution:**

If we calculate $AB$, we find that it is a linear combination of $B$:

\[
AB = \begin{bmatrix} -\frac{1}{RC_1} \left( \frac{1}{RC_1} + \frac{1}{RC_2} \right) \\ -\frac{1}{RC_2} \left( \frac{1}{RC_1} + \frac{1}{RC_2} \right) \end{bmatrix} = - \left( \frac{1}{RC_1} + \frac{1}{RC_2} \right) B
\]

This means that the controllability matrix

\[
\begin{bmatrix} B & AB \end{bmatrix}
\]

must have a rank of 1. Therefore, this system is not controllable.

(c) Explain, in terms of circuit currents and voltages why this system isn’t controllable. *(Hint: Think about what currents/voltages of the circuit we are controlling with $V_s$)*

**Solution:**

We can only control $V_s$, which in turn controls the amount of current flowing through the circuit. Since this current is equal through both capacitors and current directly affects the voltage across a capacitor, there is no way to individually controlling the voltages across the capacitors.

(d) Draw an equivalent circuit of this system that is controllable. What quantity can you control in this system?

**Solution:**

\[
\begin{tikzpicture}
\node[draw,circle] (A) at (1,0) {$V_s$};
\node[draw,circle] (B) at (0,1) {$C_3$};
\node[draw,circle] (C) at (1.5,0) {$R$};
\draw[->] (A) -- (C);
\draw[->] (B) -- (C);
\end{tikzpicture}
\]

We can control $V_3$ in this circuit.

4. **Controllability in 2D**

Consider the control of some two-dimensional linear discrete-time system

\[
\bar{x}[k+1] = A\bar{x}[k] + Bu[k]
\]

where $A$ is a $2 \times 2$ real matrix and $B$ is a $2 \times 1$ real vector.
(a) Let $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ with $a, c, d \neq 0$, and $B = \begin{bmatrix} f \\ g \end{bmatrix}$. Find a $B$ such that the system is controllable no matter what nonzero values $a, c, d$ take on, and a $B$ for which it is not controllable no matter what nonzero values are given for $a, c, d$. You may use the controllability rank test, but explain your intuition as well.

**Solution:**

With $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the system is controllable for all nonzeros $a, c, d$, because $[B \ AB] = \begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix}$, which has full rank. With $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the system is not controllable because $[B \ AB] = \begin{bmatrix} 0 & 0 \\ 1 & d \end{bmatrix}$, which only has rank equal to 1. The intuition is that, due to the zero entry in $A$, the state $x_1$ evolves autonomously, i.e., $\frac{d}{dt} x_1(t) = ax_1(t)$, hence it needs to be controlled by some input $f$. On the other hand, we can control $x_2$ via controlling $x_1$, as $\frac{d}{dt} x_2(t) = cx_1(t) + dx_2(t)$, which implies that $x_2$ can be “tuned” by manipulating $x_1$.

(b) Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $a, d \neq 0$, and $B = \begin{bmatrix} f \\ g \end{bmatrix}$ with $f, g \neq 0$. Is this system always controllable? If not, find configurations of nonzero $a, d, f, g$ that make the system uncontrollable.

**Solution:**

No, it is uncontrollable when $a = d$. In this case, the matrix is just a constant $a$ times the identity. Therefore, when you check with the controllability test, $AB$ is just a scalar multiple of $B$ and hence linearly dependent. The intuition is that the two states are inherently “coupled” as two eigenvalues are the same. Any control input can only move the states along a line hence the states cannot reach arbitrary points in $\mathbb{R}^2$.

(c) We want to see if controllability is preserved under changes of coordinates. To begin with, let $\bar{z}[k] = V^{-1} \vec{x}[k]$, write out the system equation with respect to $\bar{z}$.

**Solution:**

$\vec{x}[k] = V\bar{z}[k]$, so we have

$$V\bar{z}[k + 1] = AV\bar{z}[k] + Bu[k]$$

$$\bar{z}[k + 1] = V^{-1}AV\bar{z}[k] + V^{-1}Bu[k]$$

(d) Now show that controllability is preserved under change of coordinates. (Hint: Use the fact that $\text{rank}(MA) = \text{rank}(A)$ for any invertible matrix $M$.)

**Solution:**

The matrix whose rank needs to be tested after the coordinate change is

$$\begin{bmatrix} V^{-1}B & V^{-1}AVV^{-1}B \end{bmatrix} = \begin{bmatrix} V^{-1}B & V^{-1}AB \end{bmatrix} = V^{-1}\begin{bmatrix} B & AB \end{bmatrix},$$

which has the same rank as $[B \ AB]$, since $V$ by assumption is full rank.

5. Buoyancy

An engineer would like to deploy an autonomous communications balloon (like Project Loon’s balloons: [https://plus.google.com/+ProjectLoon/posts/PVitqyeYweY](https://plus.google.com/+ProjectLoon/posts/PVitqyeYweY)) to provide internet connectivity to a particular geographical region. To provide reliable connectivity, the balloon must hold its position over the region it services. The balloon can control its altitude ($a$) by changing its buoyancy, but it doesn’t have any engines. In order to move horizontally (horizontal position $p$), the balloon drifts on air currents.

EECS 16B, Spring 2018, Homework 6
Consulting meteorologists, the engineer has modeled the air currents around the desired balloon position (the point (0,0)) and found the flow field shown in Figure 2.

![Project Loon Balloon](image)

**Figure 2: Project Loon Balloon**

![Wind speeds](image)

**Figure 3: Wind speeds**

where the wind speed at each point is described by the equations:

\[ v_p = -20p + 20a \]
\[ v_a = -20p + 20a \]

where the velocities are in kilometers per hour and the horizontal position and altitude are in kilometers. Putting this together with the balloon’s buoyancy control, the balloon’s dynamics are described by:

\[
\begin{bmatrix}
\dot{p} \\
\dot{a}
\end{bmatrix} =
\begin{bmatrix}
-20 & 20 \\
-20 & 20
\end{bmatrix}
\begin{bmatrix}
p \\
a
\end{bmatrix} +
\begin{bmatrix}
0 \\
5
\end{bmatrix} u
\]

(a) Write the dynamics equation in controller canonical form.

**Solution:**

\[
\det \left( \lambda + 20 \begin{bmatrix}
20 & -20 \\
20 & \lambda - 20
\end{bmatrix} \right) = 0
\]
\[
\lambda^2 = 0
\]
\[
\lambda_1 = \lambda_2 = 0
\]

Since both eigenvalues are 0, the controllable canonical form is:

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]
(b) What is the matrix $T$ for the change of variables $\vec{z} = T\vec{x}$ that transforms the original $A$ and $B$ matrices into controller canonical form?

**Solution:**

\[
R_n = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 100 \\ 5 & 100 \end{bmatrix}
\]

\[
R_n^{-1} = -\frac{1}{500} \begin{bmatrix} 100 & -100 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.2 \\ 0.01 & 0 \end{bmatrix}
\]

\[
\vec{q}^T = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}
\]

\[
T = \begin{bmatrix} \vec{q}^T \\ \vec{q}^T A \end{bmatrix} = \begin{bmatrix} 0.01 & 0 \\ -0.2 & 0.2 \end{bmatrix}
\]

(c) The engineer would like the balloon to converge to $(0, 0)$ with eigenvalues $-1$ and $-1$. What should be the state feedback gains $K$ multiplying the original state vector $\vec{x}$ to achieve this behavior? Write out the expression for $u$ in terms of $p$ and $a$.

**Solution:**

With state feedback in controller canonical form, we have:

\[
\dot{\vec{q}} - \vec{B}\vec{K} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}
\]

\[
\det \left( \begin{bmatrix} \lambda & -1 \\ k_1 & \lambda + k_2 \end{bmatrix} \right) = 0
\]

\[
\lambda^2 + k_2\lambda + k_1 = 0
\]

We would like our characteristic polynomial to be:

\[
(\lambda + 1)(\lambda + 1) = 0
\]

\[
\lambda^2 + 2\lambda + 1 = 0
\]

Matching coefficients, we can solve for $k_1$ and $k_2$:

\[
k_1 = 1 \quad \text{and} \quad k_2 = 2
\]

\[
K = \vec{K}T, \quad \text{so}
\]

\[
K = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0.01 & 0 \\ -0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} -0.39 \\ 4 \end{bmatrix}
\]

\[
u = 0.39 p - 0.4 a
\]
6. CCR Circuit

Consider the circuit below driven by a current source with current $u(t)$. The output $y(t)$ is the voltage across the resistor and the state variables are the capacitor voltages as marked in the circuit diagram.

![Two Capacitor Circuit with Current Source](image)

(a) Write a state model for this circuit.

**Solution:**

From KCL:

$$u - I_1 - I_2 = 0$$

$$u - C_1 \frac{dx_1}{dt} - C_2 \frac{dx_2}{dt} = 0$$

From KVL around the capacitors and resistor:

$$-R_1 I_2 - x_2 + x_1 = 0$$

$$-R_1 C_2 \frac{dx_2}{dt} - x_2 + x_1 = 0$$

and

$$y - x_2 + x_1 = 0$$

$$y = x_1 - x_2$$

Therefore, the equations can be written in state space form as:

$$
\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1 C_1} & \frac{1}{R_1 C_2} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \\ 0 \end{bmatrix} u(t)
$$

$$y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

(b) Find all equilibrium points when $u(t) = 0$ for all $t$.

**Solution:**

Setting

$$\begin{bmatrix} \frac{1}{R_1 C_1} & \frac{1}{R_1 C_2} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we see that there is an entire subspace of equilibria when $x_1 = x_2$. 

EECS 16B, Spring 2018, Homework 6
Consider the inverted pendulum system depicted below.

To bring \( \vec{x}(t) \) to the equilibrium \( \vec{x} = 0 \) we apply \( u(t) = K \vec{x}(t) \) and obtain the closed-loop system

We now design a state feedback controller, \( u(t) = k_1 \theta(t) + k_2 \dot{\theta}(t) + k_3 \dot{y}(t) \).

(c) Determine if the system is controllable.

**Solution:**

The controllability matrix

\[
\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{RC_1C_2} \\ 0 & \frac{1}{RC_1C_2} \end{bmatrix}
\]

has full rank, so the system is controllable.

(d) **OPTIONAL:** Determine if the system is observable.

**Solution:**

The observability matrix

\[
\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \frac{1}{C_1} & \frac{1}{C_2} \\ -\frac{1}{RC_1C_2} & -\frac{1}{RC_1C_2} \end{bmatrix}
\]

has rank 1, so the system is not observable.

(e) **OPTIONAL:** If your answer to part (c) or (d) is no, explain the physical reason for lack of controllability or observability, whichever is applicable.

**Solution:**

Note that the output is identically zero at any of the equilibria in part (b). Then, there is no way to distinguish between for example the system being in configuration \( x_1 = 1, x_2 = 1 \) from \( x_1 = 0, x_2 = 0 \).

7. Inverted Pendulum on a Rolling Cart

Consider the inverted pendulum depicted below, which is placed on a rolling cart and whose equations of motion are given by:

\[
\ddot{y} = \frac{1}{m + \sin^2 \theta} \left( \frac{u}{m} + \theta^2 \ell \sin \theta - g \sin \theta \cos \theta \right)
\]

\[
\ddot{\theta} = \frac{1}{\ell (\frac{m}{m} + \sin^2 \theta)} \left( -\frac{u}{m} \cos \theta - \dot{\theta}^2 \ell \cos \theta \sin \theta + \frac{M + m}{m} g \sin \theta \right).
\]

(a) Write the state model using the variables \( x_1(t) = \theta(t) \), \( x_2(t) = \dot{\theta}(t) \), and \( x_3(t) = \dot{y}(t) \). We do not include \( y(t) \) as a state variable because we are interested in stabilizing at the point \( \theta = 0, \dot{\theta} = 0, \dot{y} = 0 \), and we are not concerned about the final value of the position \( y(t) \).
Solution:
We have

\[
\begin{align*}
    x_1 &= x_2 \\
    x_2 &= \left( \frac{1}{l \left( \frac{M}{m} + \sin^2(x_1) \right)} \right) \left( -\frac{u}{m} \cos(x_1) - x_2^2 l \cos(x_1) \sin(x_1) + \frac{M+m}{m} g \sin(x_1) \right) \triangleq f_1(x_1, x_2, x_3, u) \\
    x_3 &= \left( \frac{1}{m + x_2^2 \sin(x_1)} \right) \left( \frac{u}{m} + x_3^2 \sin(x_1) - g \sin(x_1) \cos(x_1) \right) \triangleq f_2(x_1, x_2, x_3, u)
\end{align*}
\]

(b) Linearize this model at the equilibrium \( x_1 = 0, x_2 = 0, x_3 = 0, \) and \( u = 0, \) and indicate the resulting \( A \) and \( B \) matrices.

Solution:

We can keep in mind that \( x_1 = x_2 = x_3 = 0 \) to make the derivative much easier. Since we aren’t asked to linearize about a particular input, we can linearize about \( u^* = 0. \) This is fine because \( f_2 \) and \( f_3 \) are affine (linear plus a constant term) with respect to \( u. \)

\[
\begin{align*}
    \frac{\partial f_1}{\partial x_1}(0,0,0,0) &= 0 \\
    \frac{\partial f_1}{\partial x_2}(0,0,0,0) &= 1 \\
    \frac{\partial f_1}{\partial x_3}(0,0,0,0) &= 0 \\
    \frac{\partial f_2}{\partial x_1}(0,0,0,0) &= 0 \\
    \frac{\partial f_2}{\partial x_2}(0,0,0,0) &= 0 \\
    \frac{\partial f_2}{\partial x_3}(0,0,0,0) &= 0 \\
    \frac{\partial f_3}{\partial x_1}(0,0,0,0) &= 0 \\
    \frac{\partial f_3}{\partial x_2}(0,0,0,0) &= 0 \\
    \frac{\partial f_3}{\partial x_3}(0,0,0,0) &= 0
\end{align*}
\]

And,

\[
\frac{\partial f_1}{\partial u}(0,0,0,0) = 0, \quad \frac{\partial f_2}{\partial u}(0,0,0,0) = -\frac{1}{lM}, \quad \frac{\partial f_3}{\partial u}(0,0,0,0) = \frac{1}{M}
\]

Since \( x^* = 0 \) and \( u^* = 0, \) we can use the same state variables \( x \) and \( u. \) Then,

\[
\begin{align*}
    \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{lM} & 0 & 0 \\ -\frac{m}{M} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{lM} \\ \frac{1}{M} \end{bmatrix} u \\
    &= A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + B u
\end{align*}
\]

(c) Show that the linearized model is controllable.

Solution:

Observe that

\[
AB = \begin{bmatrix} -\frac{1}{lM} \\ 0 \\ 0 \end{bmatrix}
\]

and

\[
A^2B = \begin{bmatrix} 0 \\ \frac{M+m}{lM^2} \frac{g}{G} \\ \frac{m}{lM^2} \frac{g}{G} \end{bmatrix}
\]

Then,

\[
C = \begin{bmatrix} 0 & -\frac{1}{lM} & 0 \\ -\frac{1}{lM} & 0 & -\frac{M+m}{lM^2} \frac{g}{G} \\ \frac{1}{M} & 0 & \frac{m}{lM^2} \frac{g}{G} \end{bmatrix}
\]
Since we are trying to test rank, we can remove scalar terms from the vectors. We then get,

$$\text{rank } C = \text{rank } \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ l & 0 & \left(\frac{m}{M+m}\right)l \end{bmatrix}$$

We want to show that,

$$\frac{m}{M+m} \neq 1$$

This must be the case because this is only true when \( M = 0 \), which is not possible since an object at this macro scale must have mass.

(d) Suppose \( M = 1, m = 0.1, l = 1 \), and \( g = 10 \), and design a state feedback controller,

$$u(t) = -k_1 \theta(t) - k_2 \dot{\theta}(t) - k_3 \ddot{y}(t),$$

such that the eigenvalues of \( A - BK \) (the “closed-loop eigenvalues”) are \( \lambda_1 = \lambda_2 = \lambda_3 = -1 \).

**Solution:**

Plugging in values, the system is

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} u$$

Setting \( u = -K\bar{x} \), we get

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 11 + k_1 & k_2 & k_3 \\ -k_1 - 1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic polynomial is

$$p_{(A-BK)}(\lambda) = \lambda^3 + \lambda^2 (k_3 - k_2) + \lambda (-k_1 - 11) - 10k_3 = 0$$

Our target polynomial is

$$p_{(A-BK)}(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1$$

Comparing coefficients, we get

$$k_1 = -14, k_2 = -3.1, k_3 = -0.1$$

(e) Suppose we set \( k_2 = k_3 = 0 \) and vary only \( k_1 \); that is, the controller uses only \( \theta(t) \) for feedback. Does there exist a \( k_1 \) value such that all closed-loop eigenvalues have negative real parts?

**Solution:**

The characteristic polynomial is

$$p_{(A-BK)}(\lambda) = \lambda^3 + \lambda (-k_1 - 11) = 0$$

No matter what \( k_1 \) is, there will always be an eigenvalue at 0.
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