LINEAR TIME-INVARIANT (LTI) SYSTEMS

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1. INTRODUCTION

![Diagram showing classification of systems: Linear, Time-Invariant (LTI), Time-Varying (LTV), Nonlinear, and our focus on LTI systems.]

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WHY FOCUS ON LTI SYSTEMS?

- EASIEST TO ANALYSE AND UNDERSTAND → POWERFUL, ELEGANT INSIGHTS.
- VERY USEFUL APPROXIMATION OF REAL WORLD SYSTEMS
- WITHOUT KNOWING LTI WELL, CAN'T PROGRESS TO LTV/NONLINEAR

EXAMPLES OF LTI SYSTEMS

- PRETTY MUCH EVERY SYSTEM WE HAVE DONE IN THIS CLASS!
- RC, RLC CIRCUITS
- (LINEARIZED) PENDULUM
- CO-OPERATIVE CAR CONTROL
2. **RECAP OF LINEARITY**

- **"If and only if"**

$$\Downarrow \text{LINEAR IFF : } \Downarrow \text{"LEADS TO"}$$

1. **SCALING**: If \( u(t) \rightarrow y(t) \), then \((\alpha u(t)) \rightarrow (\alpha y(t)) \) [\( \alpha u(t), \alpha y(t) \)]

2. **SUPERPOSITION**: If \( u_1(t) \rightarrow y_1(t) \), \( u_2(t) \rightarrow y_2(t) \), then

\[ (u_1(t) + u_2(t)) \rightarrow (y_1(t) + y_2(t)) \] [\( u(t), y(t) \)]

**EXAMPLE**: defecred to a little lake.

3. **TIME INVARIANCE**

- **IN WORDS**: SHIFTING THE INPUT (IN TIME) Shifts THE OUTPUT (BY THE SAME AMOUNT)

- **IN EQUATIONS**: if \( u(t) \rightarrow y(t) \), then \( u(t-\zeta) \rightarrow y(t-\zeta) \) [\( \zeta \in \mathbb{R}, \zeta u(t) \)]

(\( \zeta \) is A CONSTANT, DOES NOT DEPEND ON \( t \))

**IN PICTURES**

\[ \frac{\text{u}(t)}{u(t-\zeta)} \rightarrow \frac{\text{y}(t)}{y(t-\zeta)} \]
**Examples:**

→ **Linear but not time invariant:**

→ \( y(t) = 2 \sin(2\pi t) \ u(t) \)

→ **Linear? Check scaling & superposition: Yes**

→ \( u(t) \) try, \( u(t) = \cos(2\pi t) \)

→ \( y(t) = 2 \sin(2\pi t) \cos(2\pi t) = \sin(4\pi t) \)

\[
\begin{align*}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(2A) &= 2 \sin A \cos A
\end{align*}
\]

→ **Shift** \( u(t) \) by \( 0.5 \): \( u(t-0.5) = \cos(2\pi(t-0.5)) = \cos(2\pi t - \pi) = -\cos(2\pi t) \)

→ \( y_{\text{new}}(t) = 2 \sin(2\pi t) u(t-0.5) = -2 \sin(2\pi t) \cos(2\pi t) = -\sin(4\pi t) \)

\[
\begin{align*}
\sin(\theta) &= y(t-0.5) \\
-\sin(4\pi t) &= y(t-0.5)
\end{align*}
\]

→ If TI, then \( y_{\text{new}}(t) \) should be \( \bar{y}(t-0.5) = \sin(4\pi(t-0.5)) \)

= \sin(4\pi t - 2\pi) = \sin(4\pi t)

→ **But it is not!**

→ **Hence not TI.**

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**Example: not linear, but time invariant**

→ \( y(t) = u(t) \)

→ **Linear?** Scaling: \( u(t) = 1 \) \( \Rightarrow \) \( y(t) = 1 \) \( \Rightarrow \) \( u_{\text{new}}(t) = u(t-2) \) \( \Rightarrow \) **Not linear**

→ **TI?**

\( u_{\text{new}}(t) \) \( \Rightarrow \) \( y_{\text{new}}(t) = \bar{y}_{\text{new}}(t) = u(t-2) \)

\( y(t-2) = u(t-2) \)

\( y(t) \) \( \equiv \) \( u(t-2) \) \( \Rightarrow \) **The same \( \Rightarrow \) TI.**
LINEAR AND TIME INVARIANT (LTI)

EXAMPLE:

\[ y(t) = \frac{u(t) - y(t)}{R} \]

LINEAR?

1. SCALING: if \( u_{new}(t) = a u(t) \) and \( y_{new}(t) = a y(t) \), is the eqn. satisfied?

\[ C \cdot y_{new}(t) = \frac{u_{new}(t) - y_{new}(t)}{R} \]

\[ C \cdot y(t) = \frac{a u(t) - a y(t)}{R} \]

\[ C \cdot y(t) = \frac{a (u(t) - y(t))}{R} \]

\[ C \cdot y(t) = \frac{u(t) - y(t)}{R} \]

which is true, from defn. of \( y(t) \)

2. SUPERPOSITION: also yes (left as an exercise)

TI?

Take \( u_{new}(t) = u(t-\tau) \)

Does \( y_{new}(t) \) satisfy the system equation?

IN WORDS: \( y_{new}(t) \) is just \( y(t) \) evaluated at \( s = t-\tau \)

or, \( y(t) \) evaluated at \( t = t+\tau \)

\[ \frac{d}{dt} y_{new}(s) = \frac{d}{ds} y(s) \bigg|_{s=t-\tau} \]

\[ C \cdot \frac{d}{ds} y(s) = \frac{u(s) - y(s)}{R} \]

\[ C \cdot \frac{d}{ds} y(s) \bigg|_{s=t-\tau} = \frac{(u(s) - y(s))}{R} \bigg|_{s=t-\tau} \]

\[ C \cdot \frac{d}{ds} y(s) \bigg|_{s=t-\tau} = \frac{u(t-\tau) - y(t-\tau)}{R} = \frac{u_{new}(t) - y_{new}(t)}{R} \]

\[ \frac{d}{dt} y_{new}(t) = \frac{u_{new}(t) - y_{new}(t)}{R} \]

\[ \frac{d}{dt} y_{new}(t) = \frac{u_{new}(t) - y_{new}(t)}{R} \]

SHIFTED WAVEFORMS \( u_{new}(t) \) and \( y_{new}(t) \)

Satisfy the system equation \( \Rightarrow \) IT IS TI
ASIDE: A USEFUL WAY TO INTERPRET (DIFFERENTIAL) EQUATIONS

→ Suppose you have an equation like:

\[ \frac{C}{R} \frac{dy(t)}{dt} = \frac{u(t) - y(t)}{R} \]

, with some given input \( u(t) \)

→ Pick some \( y(t) \)

→ Plot the LHS

→ Plot the RHS

→ Do they match?
  → Yes: \( y(t) \) is a solution
  → No: \( y(t) \) is not a solution (try again)

→ Revisit time invariance proof:

1. Start with a solution: \( u(t), y(t) \)

2. \( \text{RHS} = \frac{u(t) - y(t)}{R} \)

   \[ \text{LHS} = \frac{C}{R} \frac{dy(t)}{dt} \]

3. Define \( \text{new}(t) = u(t - \tau) \)

   \( y_\text{new}(t) = y(t - \tau) \)

   \[ \text{RHS}_{\text{new}}(t) = \frac{u(t - \tau) - y(t - \tau)}{R} = \text{RHS}(t - \tau) \]

   \[ \text{LHS}_{\text{new}}(t) = \frac{C}{R} \frac{dy(t - \tau)}{dt} \rightarrow \text{Derivative of the waveform} \ y(t - \tau) \]
STANDARD LINEARIZED STATE-SPACE EQUATION IS ACTUALLY LTI

\[
\frac{d \mathbf{x}(t)}{dt} = A \mathbf{x}(t) + B \mathbf{u}(t), \quad \mathbf{y}(t) = C \mathbf{x}(t) + D \mathbf{u}(t) \quad \text{(c.t.)}
\]

\[
\mathbf{x}[k+1] = A \mathbf{x}[k] + B \mathbf{u}[k], \quad \mathbf{y}[k] = C \mathbf{x}[k] + D \mathbf{u}[k] \quad \text{(d.t.)}
\]

EXACTLY THE SAME REASONING AS FOR THE RC CIRCUIT EXAMPLE SHOWS IT IS LTI

LEFT AS EXERCISES

POINT TO PONDER ON YOUR OWN: HOW DOES THE INITIAL CONDITION FIGURE IN LINEARITY AND TIME INVARIANCE?

HINT: WORK IT OUT FIRST FOR THE RC CIRCUIT EXAMPLE.

THERE MAY BE A HW ON THIS

5. IMPULSE RESPONSES OF LTI SYSTEMS

AMAZING FACT: IF YOU KNOW AN LTI SYSTEM'S IMPULSE RESPONSE, YOU CAN CALCULATE ITS RESPONSE TO ANY INPUT

WHAT IS THE IMPULSE RESPONSE?

(NEXT PAGE)
\[ u[k] \rightarrow \text{LTI SYSTEM} \rightarrow y[k] \]

**IMPULSE RESPONSE**

\[ u[k] \rightarrow y[k] \]

\[ y[k] = h[k] \]

\[ \delta[k] = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases} \quad \text{DISCRETE-TIME IMPULSE (OR DELTA FUNCTION)} \]

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**CAUSALITY**: ANOTHER IMPORTANT SYSTEM PROPERTY

→ **IN WORDS**: THE SYSTEM'S RESPONSE CAN ONLY COME AFTER AN INPUT HAS BEEN APPLIED

→ **IN EQUATIONS**:

1) **TAKE ANY INPUT** \( u[k] \), AND CORRESPONDING OUTPUT \( y[k] \) (i.e., \( u[k] \rightarrow y[k] \))

2) **TAKE ANY NUMBER** \( \tau \)

3) **DEVISE A NEW INPUT** \( u'[k] \) **THAT MATCHES** \( u[k] \) **UP TO** \( t = \tau \), **BUT IS DIFFERENT** FOR \( t \geq \tau \)

\[ u'[k] = u[k] \quad \text{FOR} \quad t < \tau, \quad \text{DIFFERENT THEREAFTER}. \]

4) **APPLY** \( u'[k] \) **TO THE SYSTEM TO GET** \( y'[k] \): \( u'[k] \rightarrow y'[k] \)

5) **CHECK**: IS \( y'[k] = y[k] \) **FOR** \( t < \tau \)?

→ **IF YES** — **FOR ALL CHOICES** OF \( u[k] \), \( \tau \) AND \( u'[k] \) — THEN THE SYSTEM IS CAUSAL

→ **IN PICTURES**

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**INPUT**

\[ u(t) \rightarrow t \]

**CAUSAL RESPONSE**

\[ u'[t] \rightarrow t \]

**RESPONSE NOT CAUSAL**

\[ u[t] \rightarrow t \]
CHECKING A SYSTEM FOR CAUSALITY CAN BE TEDIOUS

BUT FOR LTI SYSTEMS, IT IS VERY EASY

JUST FIND THE IMPULSE RESPONSE \( h[t] \).

\[ h[t] = 0 \quad \text{if} \quad t < 0 \quad \Leftrightarrow \quad \text{CAUSAL} \]

PROOF? LEFT AS EXERCISE

CLAIM: IF YOU KNOW \( h[t] \) FOR AN LTI SYSTEM, YOU CAN CALCULATE \( y[t] \) FOR ANY GIVEN INPUT \( u[t] \).

PROOF: GIVEN ANY \( u[t] \), WE CAN WRITE IT AS A SHIFTED AND SCALED SUM OF IMPULSE FUNCTIONS

\[ u[t] = \sum_{i=-\infty}^{\infty} w[i] \delta[t-i] \]

RESPONSE OF SYSTEM TO \( \delta[t-i] \)?

BY TIME INVARIANCE: IT IS \( h[t-i] \), i.e., \( \delta[t-i] \mapsto h[t-i] \)

RESPONSE TO \( u[0] \delta[t-i] \)?

BY SCALING: \( u[0] \delta[t-i] \mapsto u[0] h[t-i] \)

RESPONSE TO \( \sum_{i=-\infty}^{\infty} u[i] h[t-i] = u[t] \)?

BY SUPERPOSITION: \( \sum_{i=-\infty}^{\infty} u[i] h[t-i] \mapsto \sum_{i=-\infty}^{\infty} u[i] h[t-i] \)

OR: IF \( u[t] \mapsto y[t] \), then \( y[t] = \sum_{i=-\infty}^{\infty} u[i] h[t-i] \)
- If the system is causal, then: \( h(t) = 0 \) if \( t < 0 \)

\[
y(t) = \sum_{i=-\infty}^{t} u[i] h(t-i) = \sum_{j=0}^{t} u[t-j] h[j]
\]

This is a discrete-time convolution

\[
y(t) = u[t] \ast h(t)
\]

- Further: if \( u[t] = 0 \) for \( t < 0 \), then

\[
y(t) = \sum_{i=0}^{t} u[i] h(t-i) = \sum_{j=0}^{t} u[t-j] h[j]
\]

- Example: \( x[t+1] = ax[t] + bu[t] \) (with zero initial condition, i.e., \( x[0] = 0 \))

\[
y(t) = cx[t] + du[t]
\]

- Impulse input: \( u(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases} \)

\[
y(0) = d \cdot u[0] = d
\]

\[
\]

\[
\]

\[
\]

\[
x[t] = a^{t-1} b \quad ; \quad y[t] = cax[t] + du[t] = ca^{t-1} b
\]

Thus \( h(t) = \begin{cases} 0, & \text{if } t = 0 \\ a^{t-1} b, & \text{if } t > 0 \end{cases} \)

- Compound interest example (from long ago): \( S[t+1] = S[t](1 + r/12) + u[t] \)

\[
y(t) = S[t]
\]

\[
a = (1 + r/12), \quad b = 1, \quad c = 1, \quad d = 0
\]

\[
h(t) = (1 + r/12)^{t-1}, \quad t > 0
\]

\[
h(t) = 0 \quad \text{otherwise}
\]

\[
\text{Unstable (but we don't mind in this case)}
\]

Another question to ponder: Can we characterize Gibbs stability/instability in terms of \( h(t) \) alone (we just know \( h(t) \) - nothing else).
\[ y[n] = u[n] \ast h[n] = \sum_{i=0}^{n} u[i] h[n-i] \quad \text{(assuming causality + } u[n<0] = 0) \]

- \[ y[0] = \sum_{i=0}^{0} u[i] h[0-i] = 3 \]
- \[ y[1] = \sum_{i=0}^{1} u[i] h[1-i] = 1 + 4 + 9 + 0 = 14 \]
- \[ y[2] = \sum_{i=0}^{2} u[i] h[2-i] = 2 + 6 + 6 = 14 \]
- \[ y[3] = \sum_{i=0}^{3} u[i] h[3-i] = 4 + 3 = 7 \]
- \[ y[4] = \sum_{i=0}^{4} u[i] h[4-i] = 2 + 6 = 8 \]
- \[ y[5] = \sum_{i=0}^{5} u[i] h[5-i] = 3 \]
- \[ y[6] \text{ and above : 0} \]
1. Keep a copy of $u(t)$ handy:

![Graphical Convolution Example]

2. Mirror $h(t)$ around $t=0$ and keep handy ($h[-t]$):

![Mirror Example]

3. To get $y(t)$:
   - 3a: Shift $h[-t]$ to the right by $t$ and place over $u(t)$.
   - 3b: Multiply and add up to get $y(t)$.

4. Repeat for every $t$.

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### 7. Impulse Response and Convolution for C.T. Systems

![Impulse Response Diagram]

→ Apply Dirac δ-function $δ(t) = \begin{cases} \infty, & t=0 \\ 0, & \text{otherwise} \end{cases}$, satisfying $\int_{-\infty}^{\infty} δ(t) \, dt = 1$, any $\epsilon>0$

→ Record output $h(t)$: this is the C.T. impulse response.

→ Then, given any $u(t)$, with $u(t) \rightarrow y(t)$, $y(t) = \int_{-\infty}^{t} u(τ) \, h(t-τ) \, dτ = u(t) \ast h(t)$

→ Proof: Analogous to D.T. case, using properties of Dirac δ.

→ Causality: $h(t) = 0$ for $t<0$ implies $y(t) = \int_{-\infty}^{t} u(τ) \, h(t-τ) \, dτ$