4. **Examples (cont'd):**

\[
\begin{align*}
\text{LHS} & = C \frac{dy(t)}{dt} \\
\text{RHS} & = \frac{u(t) - y(t)}{R}
\end{align*}
\]

→ **Aside**: A way to look at Eqs. like these.

1. Pick \( y(t) \) (\( u(t) \), the input, is given + fixed).

2. Calculate:
   - \( \text{RHS} = \frac{u(t) - y(t)}{R} \)
   - \( \text{LHS} = C \frac{dy(t)}{dt} \)

3. Check if they overlap
   - If yes: \( y(t) \) is a solution
   - No? Try again.

→ Showing that the above Eqs. are time invariant.
If I try: \( u_{\text{new}}(t) = u(t-\tau) \), \( y_{\text{new}}(t) = y(t-\tau) \)

→ then \( u_{\text{new}}(t) \) & \( y_{\text{new}}(t) \) SHOULD ALSO SOLVE

\[
\frac{dy_{\text{new}}}{dt}(t) = \frac{u(t)-y(t)}{R}
\]

\[
\text{LHS} = C \frac{dy_{\text{new}}}{dt} \text{(t)}
\]

\[
\text{RHS} = \frac{u(t)-y(t)}{R}
\]

→ STANDARD STATE SPACE ("LINEAR") SYSTEM:

\[
\frac{dx(t)}{dt} = A x(t) + B u(t), \quad y(t) = C x(t) + D u(t)
\]

→ THIS IS LINEAR AND TI

→ PROOF: STEPS IDENTICAL TO THE CRT. EXAMPLE ABOVE

→ POINT TO PONDER: HOW DO I.C.s OF STATE-SPACE SYSTEMS

→ FIGURE IN LINEARITY AND TIME INVARIANCE?

5. IMPULSE RESPONSES OF LTI SYSTEMS

→ AMAZING FACT: IF YOU KNOW \( h(t) \), YOU CAN CALCULATE \( y(t) \) FOR ANY INPUT \( u(t) \)
D.T. $\delta$ function or impulse

$$\delta(t) = \begin{cases} 1, & \text{if } t=0 \\ 0, & \text{otherwise} \end{cases}$$

**Causality**: (A property of systems)

**Words**: The system won't respond to an input before the input is applied.

**In Eqs**:

1. Take any input $\hat{u}(t)$, apply it: $\hat{u}(t) \rightarrow \hat{y}(t)$
2. Pick any number $\tau$
3. Devise a new input $u(t)$:
   - $u(t) = \hat{u}(t)$ for $t < \tau$
   - After that ($t \geq \tau$), $u(t)$ can be different from $\hat{u}(t)$
4. Apply $u(t)$ to the system: $u(t) \rightarrow y(t)$
5. Check: Is $y(t) = \hat{y}(t)$ for $t < \tau$? (For all choices of $\hat{u}, u$ and $\tau$)
   - **Yes**: Causal
   - **Else**: Not Causal

**Picture**:

- Input $\hat{u}(t) \rightarrow \hat{y}(t)$
- Causal: $y(t) \rightarrow \hat{y}(t)$
- Not Causal: $y(t) \rightarrow \hat{y}(t)$
Causality in LTI systems.

(IF \( h[t] = 0 \) \( \forall t < 0 \) \( \iff \) CAUSAL

PROOF: try it yourself.

CLAIM: For LTI systems, if you know \( h[t] \), and are given ANY input \( u[t] \), you can calculate \( y[t] \) (where \( u[t] \rightarrow y[t] \)).

PROOF:

\[ u[t] = u[t-0] * \delta(t) + u[t-1] * \delta(t-1) + u[t-2] * \delta(t-2) + u[t-1] * \delta(t-1) + \ldots \]

\[ u[t] = \sum_{i=-\infty}^{\infty} u[i] \delta(t-i) \]

Consider the system w/ input \( \delta[t-i] \)

\( \delta[t-i] \rightarrow h[t-i] \) (Time Invariance)

Try input \( u[i] \delta[t-i] \rightarrow u[i] h[t-i] \) (Scaling)

Try \( u[t] = \sum_{i=\infty}^{t} (u[i] \delta[t-i]) \rightarrow \sum_{i=-\infty}^{t} u[i] h[t-i] \)

\[ y[t] = \sum_{i=-\infty}^{t} u[i] h[t-i] \]

\( \Rightarrow y[t] = \sum_{j=0}^{t-i} u[t-j] h[j] \)

\( \Rightarrow \) written as \( y[t] = u[t] \ast h[t] = h[t] \ast u[t] \)

(SUPPOSE CAUSAL: \( \Rightarrow h[t] = 0 \) \( \forall t < 0 \)

\[ y[t] = \sum_{i=\infty}^{t} u[i] h[t-i] = \sum_{j=0}^{\infty} u[t-j] h[j] \]
Suppose (for convenience in problems) \( u(t) = 0 \ \forall \ t < 0 \)

\[
y(t) = \sum_{i=0}^{\infty} u[i]h[t-i] = \sum_{j=0}^{\infty} u[t-j]h[j]
\]