1 Complex Numbers

A complex number $z$ is an ordered pair $(x, y)$, where $x$ and $y$ are real numbers, written as $z = x + jy$ where $j = \sqrt{-1}$. A complex number can also be written in polar form as follows:

$$z = |z|e^{j\theta}$$

The magnitude $z$ is denoted as $|z|$ and is given by

$$|z| = \sqrt{x^2 + y^2}.$$ 

The phase or argument of a complex number is denoted as $\theta$ and is given by

$$\theta = \text{atan2}(y, x).$$

Here, $\text{atan2}(y, x)$ is a function that returns the angle from the positive x-axis to the vector from the origin to the point $(x, y)$.\(^1\)

The complex conjugate of a complex number $z$ is denoted by $\bar{z}$ (or might also be written $z^*$) and is given by

$$\bar{z} = x - jy.$$ 

Euler’s Identity is

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

---

1See its relation to $\tan^{-1}\left(\frac{y}{x}\right)$ at [https://en.wikipedia.org/wiki/Atan2](https://en.wikipedia.org/wiki/Atan2)
With this definition, the polar representation of a complex number will make more sense. Note that

\[ |z|e^{j\theta} = |z|\cos(\theta) + j|z|\sin(\theta). \]

The reason for these definitions is to exploit the geometric interpretation of complex numbers, as illustrated in Figure 1, in which case |z| is the magnitude and \(e^{j\theta}\) is the unit vector that defines the direction.

2 Useful Identities

**Complex Number Properties**

**Rectangular vs. polar forms:** \(z = x + jy = |z|e^{j\theta}\)

where \(|z| = \sqrt{x^2 + y^2}, \theta = \text{atan2}(y,x)\). We can also write \(x = |z|\cos\theta, y = |z|\sin\theta\).

**Euler’s identity:** \(e^{j\theta} = \cos \theta + j\sin \theta\)

\(\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}\)

**Complex conjugate:** \(\bar{z} = x - jy = |z|e^{-j\theta}\)

\(|z + w| = |\bar{z} + \bar{w}|, (z - w) = \bar{z} - \bar{w}\)

\(|zw| = |\bar{z}\bar{w}|, (z/w) = \bar{z}/\bar{w}\)

\(\bar{\bar{z}} = z \iff z \text{ is real}\)

\(|z^n| = (|z|)^n\)

**Complex Algebra**

Let \(z_1 = x_1 + jy_1 = |z_1|e^{j\theta_1}, z_2 = x_2 + jy_2 = |z_2|e^{j\theta_2}\).

(Note that we adopt the easier representation between rectangular form and polar form.)

**Addition:** \(z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)\)

**Multiplication:** \(z_1z_2 = |z_1||z_2|e^{j(\theta_1 + \theta_2)}\)

**Division:** \(\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\theta_1 - \theta_2)}\)

**Power:** \(z_1^n = |z_1|^n e^{jn\theta_1}\)

\(\bar{z}_1^n = \pm |z_1|^n e^{j\frac{\theta_1}{2}}\)

**Useful Relations**

\(-1 = j^2 = e^{j\pi} = e^{-j\pi}\)

\(j = e^{j\frac{\pi}{2}} = \sqrt{-1}\)

\(-j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}}\)

\(\sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\frac{\pi}{4}} = \frac{\pm(1 + j)}{\sqrt{2}}\)

\(\sqrt{-j} = (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{-j\frac{\pi}{4}} = \frac{\pm(1 - j)}{\sqrt{2}}\)

3 Phasors

We consider sinusoidal voltages and currents of a specific form:

\[
\begin{array}{c|c}
\text{Voltage} & v(t) = V_0 \cos(\omega t + \phi_v) \\
\text{Current} & i(t) = I_0 \cos(\omega t + \phi_i),
\end{array}
\]

where,

(a) \(V_0\) is the voltage amplitude and is the highest value of voltage \(v(t)\) will attain at any time. Similarly, \(I_0\) is the current amplitude.

(b) \(\omega\) is the angular frequency of oscillation. \(\omega\) is related to frequency by \(\omega = 2\pi f\). Frequency \(f\) is the number of oscillation cycles that occur per second. If \(T\) is the period of the sinusoid (that is, the amount of time it takes for one complete cycle to occur) the frequency is \(f = \frac{1}{T}\).
(c) $\phi_v$ and $\phi_i$ are the phase terms of the voltage and current respectively. These capture a delay, or a shift in time, of the sinusoid.

We know from Euler’s identity that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Using this identity, we can obtain an expression for $\cos(\theta)$ in terms of an exponential:

$$\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$$

Extending this to our voltage signal from above:

$$v(t) = V_0 \cos(\omega t + \phi_v) = \frac{V_0}{2}e^{j\omega t} + \frac{V_0}{2}e^{-j\omega t - j\phi_v} = \frac{V_0}{2}e^{j\phi_v}e^{j\omega t} + \frac{V_0}{2}e^{-j\phi_v}e^{-j\omega t}$$  \hspace{1cm} (1)

The coefficient of the $e^{j\omega t}$ term in Equation 1 is called the phasor form of this signal:

$$\tilde{V} = \frac{1}{2}V_0 e^{j\phi_v}$$

The complex conjugate of the phasor is the coefficient of the $e^{-j\omega t}$ term in Equation 1 and is given by:

$$\bar{V} = \frac{1}{2}V_0 e^{-j\phi_v}$$

The phasor representation contains the magnitude and phase of the signal, but not the time-varying portion. Phasors let us handle sinusoidal signals much more easily. They are powerful because they let us use DC-like circuit analysis techniques, which we already know, to analyze circuits with sinusoidal voltages and currents.

Note: We can only use phasors if we know that all of our signals are sinusoids.

Within this standard form, the phasor domain representation is as follows. The general equation that relates cosines to phasors is below, where $\tilde{V}$ is the phasor.

$$V_0 \cos(\omega t + \phi_v) = \tilde{V} e^{j\omega t} + \bar{V} e^{-j\omega t}$$

The standard forms for voltage and current phasors are given below:

| Voltage | $\tilde{V} = \frac{1}{2}V_0 e^{j\phi_v}$ |
| Current | $\bar{I} = \frac{1}{2}I_0 e^{j\phi_i}$ |

We define the impedance of a circuit component to be $Z = \frac{\tilde{V}}{\bar{I}}$, where $\tilde{V}$ and $\bar{I}$ represent the voltage across and the current through the component, respectively.
3.1 Phasor Relationship for Resistors

Consider a simple resistor circuit as in Figure 2 with current being

$$i(t) = I_0 \cos(\omega t + \phi)$$

By Ohm’s law,

$$v(t) = i(t)R = I_0 R \cos(\omega t + \phi)$$

In the phasor domain,

$$\tilde{V} = R \tilde{I}.$$  

We usually refer to the impedance of the resistor, $Z_R$, in the phasor domain. Since $Z_R = R$, we can also write:

$$\tilde{V} = Z_R \tilde{I}.$$  

3.2 Phasor Relationship for Capacitors

Consider a capacitor circuit as in Figure 3 with voltage being

$$v(t) = V_0 \cos(\omega t + \phi)$$
By the capacitor equation,

\[ i(t) = C \frac{dv}{dt}(t) \]
\[ = -CV_0 \omega \sin(\omega t + \phi) \]
\[ = -CV_0 \omega \left( -\cos\left( \omega t + \phi + \frac{\pi}{2} \right) \right) \]
\[ = CV_0 \omega \cos\left( \omega t + \phi + \frac{\pi}{2} \right) \]
\[ = (\omega C)V_0 \cos\left( \omega t + \phi + \frac{\pi}{2} \right) \]

In the phasor domain,

\[ \tilde{I} = \omega Ce^{j\frac{\pi}{2}} \tilde{V} = j\omega C \tilde{V} . \]

The impedance of a capacitor is an abstraction to model the capacitor in a manner similar to a resistor in the phasor domain. This is denoted \( Z_C \), which is given by:

\[ Z_C = \frac{\tilde{V}}{\tilde{I}} = \frac{1}{j\omega C} \]

1. **Inductor Impedance**

   Given the voltage-current relationship of an inductor \( V = L \frac{di}{dt} \), show that its complex impedance is \( Z_L = j\omega L \).

   **Answer:**

   ![Inductor Circuit](image)

   Consider a simple resistor circuit as in Figure 4 with current being

   \[ i(t) = I_0 \cos(\omega t + \phi) \]

   By the inductor equation,

   \[ v(t) = L \frac{di}{dt}(t) \]
   \[ = -LI_0 \omega \sin(\omega t + \phi) \]
   \[ = LI_0 \omega \cos\left( \omega t + \phi + \frac{\pi}{2} \right) \]
   \[ = (\omega L)I_0 \cos\left( \omega t + \phi + \frac{\pi}{2} \right) \]
In the phasor domain,
\[ \tilde{V} = \omega L e^{j\frac{\pi}{2}} \tilde{I} = j \omega L \tilde{I} \]

The impedance of an inductor is an abstraction to model the inductor as a resistor in the phasor domain. This is denoted \( Z_L \).

\[ Z_L = \frac{\tilde{V}}{\tilde{I}} = j \omega L \]

2. Phasor Analysis

Any sinusoidal time-varying function \( x(t) \), representing a voltage or a current, can be expressed in the form

\[ x(t) = \tilde{X} e^{j\omega t} + \tilde{\bar{X}} e^{-j\omega t}, \tag{2} \]

where \( \tilde{X} \) is a time-independent function called the phasor representation of \( x(t) \) (recall that \( \bar{a} \) denotes the complex conjugate of \( a \)). Note that 1) \( \tilde{X} \) and \( \tilde{\bar{X}} \) are complex conjugates of each other, 2) \( e^{j\omega t} \) and \( e^{-j\omega t} \) are complex conjugates of each other, and 3) that \( \tilde{X} e^{j\omega t} \) and \( \tilde{\bar{X}} e^{-j\omega t} \) are also complex conjugates of each other.

The phasor analysis method consists of five steps. Consider the RC circuit below.

![RC Circuit Diagram]

The voltage source is given by

\[ v_s(t) = 12 \sin \left( \omega t - \frac{\pi}{4} \right), \tag{3} \]

with \( \omega = 1 \times 10^3 \text{ rad/s} \), \( R = \sqrt{3} \text{k} \Omega \), and \( C = 1 \text{ \mu F} \).

Our goal is to obtain a solution for \( i(t) \) with the sinusoidal voltage source \( v_s(t) \).

(a) **Step 1: Adopt cosine references (sine references, as shown in class, also work fine)**

All voltages and currents with known sinusoidal functions should be expressed in the standard cosine format. Convert \( v_s(t) \) into a cosine and write down its phasor representation \( \tilde{V}_s \).

**Answer:**

\[ v_s(t) = 12 \cos \left( \omega t - \frac{\pi}{4} - \frac{\pi}{2} \right) = 12 \cos \left( \omega t - \frac{3\pi}{4} \right) \]

The phasor is given by

\[ \tilde{V}_s = \frac{1}{2} 12 e^{-j\frac{3\pi}{4}} = 6 e^{-j\frac{3\pi}{4}} \]
Step 2: Transform circuits to phasor domain

The voltage source is represented by its phasor $\tilde{V}_s$. The current $i(t)$ is related to its phasor counterpart $\tilde{I}$ by

$$i(t) = \tilde{I} e^{j\omega t} + \bar{\tilde{I}} e^{-j\omega t}.$$ 

What are the impedances of the resistor, $Z_R$, and capacitor, $Z_C$? We sometimes also refer to this as the "phasor representation" of $R$ and $C$.

Answer:

$$Z_R = R$$
$$Z_C = \frac{1}{j\omega C}$$

Step 3: Cast KCL and/or KVL equations in phasor domain

Use Kirchhoff’s laws to write down a loop equation that relates all phasors in Step 2.

Answer:

The KVL equations for this circuit are:

$$v_s(t) = v_R(t) + v_C(t),$$

where we have denoted the voltage across the resistor as $v_R(t)$. If you expand all these quantities using phasors (using Equation (2)), we get

$$\tilde{V}_s e^{j\omega t} + \bar{\tilde{V}}_s e^{-j\omega t} = \tilde{V}_R e^{j\omega t} + \bar{\tilde{V}}_R e^{-j\omega t} + \tilde{V}_C e^{j\omega t} + \bar{\tilde{V}}_C e^{-j\omega t}.$$ 

Collecting together all the $e^{j\omega t}$ terms and all the $e^{-j\omega t}$ terms, the above equation can be rewritten as

$$(\tilde{V}_s - \tilde{V}_R - \tilde{V}_C) e^{j\omega t} + (\bar{\tilde{V}}_s - \bar{\tilde{V}}_R - \bar{\tilde{V}}_C) e^{-j\omega t} = 0.$$ 

If $\omega \neq 0$, it can be shown that both of the coefficients of $e^{j\omega t}$ and $e^{-j\omega t}$ in the above equation must be equal to 0 for this equation to hold. That is:

$$\tilde{V}_s - \tilde{V}_R - \tilde{V}_C = 0,$$
$$\bar{\tilde{V}}_s - \bar{\tilde{V}}_R - \bar{\tilde{V}}_C = 0.$$ 

Both of the equations above have the same meaning, i.e., $\tilde{V}_s - \tilde{V}_R - \tilde{V}_C = 0$ or

$$\tilde{V}_s = \tilde{V}_R + \tilde{V}_C.$$ 

This is exactly the same KVL equation as given by $v_s(t) = v_R(t) + v_C(t)$, but using phasors. In the same way, you can show that KCL is also obeyed by phasors. This conclusion implies that the standard rules for putting together circuit equations using KCL and KVL work identically with phasors as with time-varying notation. The only difference is that the voltage-current relationships of elements should be in phasor form, e.g., $I = j\omega C V_C = \frac{1}{Z_C} \tilde{V}_C$.

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2 Try working out the $\omega = 0$ case by yourself! It’s even easier.

3 This can be shown because the functions $e^{j\omega t}$ and $e^{-j\omega t}$ are linearly independent.
We can apply what we’ve found above to write the circuit in the phasor domain:

\[ Z_R \tilde{I} + Z_C \tilde{I} = \tilde{V}_s \]

\[ \left( R + \frac{1}{j\omega C} \right) \tilde{I} = 6e^{-j\frac{\pi}{2}} \]

Now that we’ve shown that the phasor representation (i.e., \( \tilde{I} \) and \( \tilde{V}_C \)) of our circuit is equivalent to the time-varying representation (i.e., \( i(t) \) and \( v(t) \)), in the future we can write the KCL and KVL equations in phasor form directly.

(d) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for \( \tilde{I} \) and \( \tilde{V}_C \). What is the polar form of \( \tilde{I} \) and \( \tilde{V}_C \)? Polar form is given by \( Ae^{j\theta} \), where \( A \) is a positive real number.

**Answer:**

\[ I = \frac{6e^{-j\frac{\pi}{4}}}{R + \frac{1}{j\omega C}} = \frac{j6\omega Ce^{-j\frac{\pi}{4}}}{j\omega RC + 1} \]

\[ \tilde{V}_C = \tilde{I}Z_C = \frac{j6\omega Ce^{-j\frac{\pi}{4}}}{1 + j\omega RC} \cdot \frac{1}{j\omega C} = \frac{6e^{-j\frac{\pi}{4}}}{1 + j\omega RC} \]

To derive the polar form, we plug in for the values of the circuit elements:

\[ \tilde{I} = \frac{j6\omega Ce^{-j\frac{\pi}{4}}}{1 + j\omega RC} = \frac{j6 \cdot 10^3 \cdot 10^{-6} \cdot e^{-j\frac{\pi}{4}}}{1 + j \cdot 10^3 \cdot \sqrt{3} \cdot 10^3 \cdot 10^{-6}} = \frac{\left( e^{j\frac{\pi}{4}} \right) \cdot 6 \cdot 10^{-3} \cdot e^{-j\frac{\pi}{4}}}{1 + j \sqrt{3}} \]

\[ = \frac{6 \cdot 10^{-3} \cdot e^{-j\frac{\pi}{4}} e^{j\frac{\pi}{2}}}{2e^{j\frac{\pi}{2}}} = 3e^{-j\frac{\pi}{12}} mA. \]

\[ \tilde{V}_C = \frac{6e^{-j\frac{\pi}{4}}}{1 + j\omega RC} = \frac{6e^{-j\frac{\pi}{4}}}{1 + j \sqrt{3}} \]

(e) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is \( i(t) \) and \( v_C(t) \)? What is the phase difference between \( i(t) \) and \( v_C(t) \)?

**Answer:**

\[ i(t) = \tilde{I}e^{j\omega t} + \tilde{I}e^{-j\omega t} = 3e^{-j\frac{13\pi}{12}} e^{j\omega t} + 3e^{j\frac{13\pi}{12}} e^{-j\omega t} = 6\cos \left( \omega t - \frac{7\pi}{12} \right) mA \]

\[ v_C(t) = \tilde{V}_Ce^{j\omega t} + \tilde{V}_C e^{-j\omega t} = 3e^{-j\frac{13\pi}{12}} e^{j\omega t} + 3e^{j\frac{13\pi}{12}} e^{-j\omega t} = 6\cos \left( \omega t - \frac{13\pi}{12} \right) V \]

The phase difference between the \( v_c(t) \) and \( i(t) \) with respect to \( i(t) \) is \( -\frac{\pi}{2} \).

3. **RLC Circuit Phasor Analysis**

We study a simple RLC circuit with an AC voltage source given by

\[ v_s(t) = B \cos(\omega t - \phi) \]
(a) Write out the phasor representations of $v_s(t)$, $R$, $C$, and $L$.

   **Answer:**
   \[
   \tilde{V}_s = \frac{B}{2} e^{-j\phi}, \quad Z_R = R, \quad Z_C = \frac{1}{j\omega C}, \quad Z_L = j\omega L
   \]

(b) Use Kirchhoff’s laws to write down a loop equation relating the phasors in the previous part.

   **Answer:**
   \[
   Z_R \tilde{I} + Z_C \tilde{I} + Z_L \tilde{I} = \tilde{V}_s
   \]
   \[
   \left(R + \frac{1}{j\omega C} + j\omega L\right) \tilde{I} = \frac{B}{2} e^{-j\phi}
   \]

(c) Solve the equation in the previous step for the current $\tilde{I}$. What is the magnitude and phase of the polar form of $\tilde{I}$?

   **Answer:**
   \[
   \tilde{I} = \frac{\frac{B}{2} e^{-j\phi}}{R + \frac{1}{j\omega C} + j\omega L} = \frac{\frac{B}{2} e^{-j\phi}}{R + j\left(-\frac{1}{\omega C} + \omega L\right)}
   \]

   The magnitude of $\tilde{I}$ is
   \[
   |\tilde{I}| = \frac{\frac{B}{2}}{\sqrt{R^2 + \left(-\frac{1}{\omega C} + \omega L\right)^2}}
   \]

   The phase of $\tilde{I}$ is
   \[
   \angle \tilde{I} = -\phi - \tan^{-1} \left(\frac{2\omega L}{\omega^2 - \frac{1}{\omega C}}\right)
   \]

4. **(Practice) Complex Algebra**

   (a) Express the following values in polar forms: $-1$, $j$, $-j$, $\sqrt{j}$, and $\sqrt{-j}$.

   **Answer:**
The answers are also found in the table:

\[
-1 = j^2 = e^{j\pi} = e^{-j\pi} \\
j = e^{j\frac{\pi}{2}} = \sqrt{-1} \\
-j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}} \\
\sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\frac{\pi}{4}} = \pm \frac{1+j}{\sqrt{2}} \\
\sqrt{-j} = (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{-j\frac{\pi}{4}} = \pm \frac{1-j}{\sqrt{2}}
\]

(b) Represent \( \sin \theta \) and \( \cos \theta \) using complex numbers. \((\text{Hint: Use Euler's identity.})\)

**Answer:**

\[
\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \quad \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}.
\]

(c) For complex number \( z = x + jy \) how that \( |z| = \sqrt{z\bar{z}} \), where \( \bar{z} \) is the complex conjugate of \( z \).

**Answer:**

We can follow the definition of complex conjugate and magnitude:

\[
\sqrt{z\bar{z}} = \sqrt{(x + jy)(x - jy)} = \sqrt{x^2 + y^2} = |z|
\]

For the next two parts, consider two complex numbers \( V = 3 - j4 \) and \( I = -(2 + j3) \).

(d) Express \( V \) and \( I \) in polar form.

**Answer:**

We can draw out \( V \) and \( I \) in the complex plane. Then,

\[
|V| = \sqrt{VV} = 5 \\
\theta_V = \tan^{-1}\left(-\frac{4}{3}\right) = -0.927 \text{ rad} \\
V = |V|e^{j\theta_V} = 5e^{-j0.927}
\]

and similarly for \( I \):

\[
|I| = \sqrt{II} = \sqrt{13} \\
\theta_I = \tan^{-1}\left(\frac{3}{2}\right) - \pi \text{ rad} = -2.159 \text{ rad} \\
I = |I|e^{j\theta_I} = \sqrt{13}e^{-j2.159}
\]

(e) Find \( VI, V\bar{I}, \frac{V}{I} \), and \( \sqrt{I} \).

**Answer:**

We follow the complex algebra:

\[
VI = (5e^{-j0.927})(\sqrt{13}e^{-j2.159}) = 5\sqrt{13}e^{-j3.086}
\]
\[ VI = (5e^{-j0.927})(\sqrt{13}e^{j2.159}) = 5\sqrt{13}e^{j1.232} \]

\[ \frac{V}{I} = \frac{5e^{-j0.927}}{\sqrt{13}e^{-j2.159}} = \frac{5\sqrt{13}}{13}e^{j1.232} \]

\[ \sqrt{I} = \sqrt{\sqrt{13}e^{-j2.159}} = (13^{\frac{1}{3}})e^{-j\frac{2.159}{4}} = \pm \frac{13^{\frac{1}{4}}}{3}e^{-j1.0795} \]

(f) For complex number \( z = x + jy \), what are the roots of \( z^2 = 1 \)? What about \( z^3 = 1 \)? How many roots does \( z^n = 1 \) have? What is the general form for the solutions of \( z^n = 1 \)?

**Answer:**

For the roots of \( z^2 = 1 \), \( z^2 \) is a standard quadratic, so we have \( z = 1, -1 \) by solving the quadratic equation. To solve \( z^3 = 1 \), we substitute \( 1 = e^{j0}, e^{j2\pi}, e^{j4\pi}, \ldots \), which are the complex representations of 1. For each complex representation of 1:

\[ z^3 = e^{j0}, e^{j2\pi}, e^{j4\pi} \]

Then take the cube root for each case:

\[ z = e^{j\frac{0}{3}}, e^{j\frac{2\pi}{3}}, e^{j\frac{4\pi}{3}} \]

Notice that for \( e^{j6\pi} \) and greater, the solutions repeat the above three solutions.

The general solution for \( z^n = 1 \) is

\[ z = e^{j2\pi k/n} \]

for \( 0 \leq k < n \).

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