Properties of Orthonormal Vectors

Please refer to the lecture 10A notes online for proofs of the following properties.

(a) **Definition: Orthonormal**

A set of vectors \( \{ \vec{v}_1, \ldots, \vec{v}_n \} \) is **orthonormal** if all the vectors are mutually orthogonal to each other and all are of unit length. That is:

- **Orthogonal:** For all pairs of vectors \( \vec{v}_i, \vec{v}_j \) where \( i \neq j \), \( \langle \vec{v}_i, \vec{v}_j \rangle = 0 \). For real vectors, this means \( \vec{v}_i^T \vec{v}_j = 0 \).
- **Normalized:** For all \( i \), \( \| \vec{v}_i \| = 1 \). (This implies that \( \| \vec{v}_i \| = \langle \vec{v}_i, \vec{v}_i \rangle = 1 \).)

(b) Any set of orthogonal (and by extension orthonormal) vectors are linearly independent.

(c) Any set of orthogonal (and orthonormal) vectors \( \{ \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n \} \) form a basis in \( \mathbb{R}^n \).

Let us consider the orthonormal set of vectors \( \{ \vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n \} \), which form a basis in \( \mathbb{R}^n \). For any vector \( \vec{v} \) represented in this basis, we have

\[
\vec{v} = \alpha_1 \vec{q}_1 + \alpha_2 \vec{q}_2 + \cdots + \alpha_n \vec{q}_n.
\]

Hence, \( \alpha_i = \vec{q}_i^T \vec{v} \). More compactly, we can write

\[
\vec{v} = Q^T \vec{v}, \quad \vec{v}, \vec{\alpha}, \{ \vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n \}, \text{ and } Q \text{ have the following properties.}
\]

(d) The vector of projections, \( \vec{\alpha} \), has the same norm as the original vector \( \vec{v} \).

(e) Given that the columns of \( Q \) are orthonormal, the rows of \( Q \) are also orthonormal.

(f) Given a set of orthonormal vectors \( \{ \vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n \} \), for any \( r \leq n \), we have

\[
||\vec{q}_1||^2 + ||\vec{q}_2||^2 + \cdots + ||\vec{q}_r||^2 = \sum_{i=1}^{r} \sum_{j=1}^{n} q_{ji}^2 = r.
\]

Here, \( q_{ji} \) is the \( j^{th} \) element of \( \vec{q}_i \).
Properties of Real Symmetric Matrices

Let $T$ be a symmetric matrix on $\mathbb{R}^{n \times n}$. Then,

(a) The eigenvalues of $T$ are real.

(b) A set of real eigenvectors $\{\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\}$ of $T$ can be found.

(c) The eigenvectors $\{\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\}$ of $T$ are orthogonal.

(d) A real orthonormal set of eigenvectors of $T$ can be found by normalizing each vector in the set $\{\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\}$.

That is, there exists $n$ real eigenvalues and $n$ real linearly independent eigenvectors of $T$ that form a basis for $\mathbb{R}^n$. Furthermore, these eigenvectors can be normalized to make an orthonormal basis.

Outer Products

We can define an outer product between two vectors $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$ as follows:

$$\vec{x}\vec{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}_{1 \times m} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_m \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_m \end{bmatrix}_{n \times m}$$

Hence, the outer product gives us an $n \times m$ rank-1 matrix. Note: Do not confuse the outer product $\vec{x}\vec{y}^T$ with the inner product given by $\vec{x}^T \vec{y}$.

We can represent the matrix multiplication as a sum of outer products as follows,

$$XY^T = \begin{bmatrix} \vec{x}_1^T \\ \vec{x}_2^T \\ \vdots \\ \vec{x}_r^T \end{bmatrix} \begin{bmatrix} \vec{y}_1^T & \vec{y}_2^T & \cdots & \vec{y}_r^T \end{bmatrix} = \sum_{i=1}^{r} \vec{x}_i \vec{y}_i^T.$$

Where $X$ and $Y^T$ are any $n \times r$ and $r \times m$ matrices.

Singular Value Decomposition

The SVD is a useful way to characterize a matrix. Let $A$ be a matrix from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or $A \in \mathbb{R}^{m \times n}$) of rank $r$. It can be decomposed into a sum of $r$ rank-1 matrices:

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

where

- $\vec{u}_1, \ldots, \vec{u}_r$ are orthonormal vectors in $\mathbb{R}^m$; $\vec{v}_1, \ldots, \vec{v}_r$ are orthonormal vectors in $\mathbb{R}^n$.
• the singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ are always real and positive.

We can rewrite the decomposition in the form

$$A = U_1 S V_1^T,$$

where

• $U_1$ is an $[m \times r]$ matrix whose columns consist of $\vec{u}_1, \ldots, \vec{u}_r$ (orthonormal vectors in $\mathbb{R}^m$). Consequently,

$$U_1^T U_1 = I_{r \times r}$$

• $V_1$ is an $[n \times r]$ matrix whose columns consist of $\vec{v}_1, \ldots, \vec{v}_r$ (orthonormal vectors in $\mathbb{R}^n$). Consequently,

$$V_1^T V_1 = I_{r \times r}$$

• $U_1$ characterizes the column space of $A$ and $V_1$ characterizes the row space of $A$.

• $S$ is an $[r \times r]$ matrix whose diagonal entries are the singular values of $A$ arranged in descending order. The singular values are the square roots of the nonzero eigenvalues of $A^T A$ (or, identically, $AA^T$).

The full matrix form of SVD is

$$A = U \Sigma V^T$$

where $U^T U = I_{m \times m}, V^T V = I_{n \times n}, \Sigma \in \mathbb{R}^{m \times n}$, which contains $S$ and elsewhere zero.

Questions

1. Eigenvectors are Orthogonal

Prove the following: For any symmetric matrix $A$, any two eigenvectors corresponding to distinct eigenvalues of $A$ are orthogonal.

Hint: Use the definition of an eigenvalue to show that $\lambda_1 (\vec{v}_1^T \vec{v}_2) = \lambda_2 (\vec{v}_1^T \vec{v}_2)$.

2. Frobenius Norm

In this problem we will investigate the properties of the Frobenius norm.

Much like the norm of a vector $\vec{x} \in \mathbb{R}^N$ is $\|x\| = \sqrt{\sum_{i=1}^{N} x_i^2}$, the Frobenius norm of a matrix $A \in \mathbb{R}^{N \times N}$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} |A_{ij}|^2}.$$ 

Note that matrices have other types of norms as well.
(a) With the above definitions, show that,

\[ \|A\|_F = \sqrt{\text{Tr}\{A^T A\}}. \]

*Note:* The trace of a matrix is the sum of its diagonal entries. For example, let \( A \in \mathbb{R}^{N \times N} \), then,

\[ \text{Tr}\{A\} = \sum_{i=1}^{N} A_{ii} \]

(b) Show that if \( U \) and \( V \) are orthonormal matrices, then

\[ \|UA\|_F = \|AV\|_F = \|A\|_F. \]

(c) Show that \( \|A\|_F = \sqrt{\sum_{i=1}^{N} \sigma_i^2} \), where \( \sigma_1, \ldots, \sigma_N \) are the singular values of \( A \).

3. **SVD Short Questions**

Assume we have the compact form of the SVD of

\[ A = U_1 S V_1^T = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_iT. \]

(a) Compute \( AV_1V_1^T \).

(b) What is the subspace that spans the column space of \( A \)?

4. **(Optional) Symmetric Matrix Properties**

Given the symmetric matrix

\[ T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \]

find the eigenvalues and eigenvectors of \( T \).

Show that the eigenvalues are real and the eigenvectors are orthonormal.

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