1. Changing Coordinates and Systems of Differential Equations

Suppose we have the pair of differential equations (valid for \( t \geq 0 \))

\[
\frac{d}{dt} x_1(t) = -5x_1(t) \quad (1)
\]

\[
\frac{d}{dt} x_2(t) = -2x_2(t) \quad (2)
\]

with initial conditions \( x_1(0) = 1 \) and \( x_2(0) = 2 \).

(a) Solve for \( x_1(t) \) and \( x_2(t) \) for \( t \geq 0 \).

(b) Suppose we want to change variables to:

\[
\tilde{x}_1(t) = x_1(t) + x_2(t)
\]

\[
\tilde{x}_2(t) = -x_1(t) + 2x_2(t)
\]

Write out a system of differential equations relating the \( \frac{d}{dt} \tilde{x}_i(t) \) to the \( \tilde{x}_i(t) \). The system can have equations that involve both \( i = 1, 2 \) variables on the right hand side, but should only involve one of the \( \frac{d}{dt} \tilde{x}_i(t) \) on each of the left hand sides.

Change to these new coordinates using two methods.

**Method 1: Direct Substitution** Solve for \( \tilde{x}_i(t) \) by direct substitution from \( x_i(t) \).

**Method 2: Linear Algebra Change-of-Coordinates** Solve for \( \tilde{x}_i(t) \) using a change of coordinates, like you saw in EE16A.

(c) How do the initial conditions for \( x_i(t) \) translate into the initial conditions for \( \tilde{x}_i(t) \)? What are the solutions for \( \tilde{x}_i(t) \)?

(d) Consider a "new" system of differential equations (valid for \( t \geq 0 \))

\[
\frac{d}{dt} y_1(t) = -4y_1(t) + y_2(t) \quad (3)
\]

\[
\frac{d}{dt} y_2(t) = 2y_1(t) - 3y_2(t) \quad (4)
\]

with initial conditions \( y_1(0) = 3 \) and \( y_2(0) = 3 \).

Write out the differential equations and initial conditions in matrix/vector form.

(e) Find the eigenvalues \( \lambda_1, \lambda_2 \) and eigenspaces for the differential equation matrix above.

(f) Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables \( z_{\lambda_1}(t), z_{\lambda_2}(t) \). (These variables represent eigenbasis-aligned coordinates.)

(g) Solve the differential equation for \( z_{\lambda_i}(t) \) in the eigenbasis.

(h) Convert your solution back into the original coordinates to find \( y_i(t) \).
(i) We can solve this equation using a slightly shorter approach by observing that the solutions for $y_i(t)$ will all be of the form

$$y_i(t) = \sum_k K_{ik} e^{\lambda_k t}$$

where $\lambda_k$ is an eigenvalue of our differential equation relation matrix and the $K_{ik}$ are constants derived from our initial conditions and the coordinate changes involved.

Since we have observed that the solutions will include $e^{\lambda_k t}$ terms, once we have found the eigenvalues for our differential equation matrix, we can guess the forms of the $y_i(t)$ as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t} \\ \gamma e^{\lambda_1 t} + \kappa e^{\lambda_2 t} \end{bmatrix}$$

where $\alpha, \beta, \gamma, \kappa$ are all constants.

Take the derivative to write out

$$\begin{bmatrix} \frac{d}{dt} y_1(t) \\ \frac{d}{dt} y_2(t) \end{bmatrix}.$$ 

and connect this to the given differential equation. Solve for $y_i(t)$ from this form of the derivative.

2. Differential Equations with Complex Eigenvalues

Suppose we have the pair of differential equations

$$\frac{d}{dt} x_1(t) = \lambda x_1(t)$$
$$\frac{d}{dt} x_2(t) = \overline{\lambda} x_2(t)$$

with initial conditions $x_1(0) = c_0$ and $x_2(0) = \overline{c_0}$, where $\lambda$ and $c_0$ are complex numbers and $\overline{\lambda}$ and $\overline{c_0}$ are their complex conjugates, respectively.

Suppose now that we have the following different variables related to the original ones:

$$\dot{x}_1(t) = ax_1(t) + \overline{a} x_2(t)$$
$$\dot{x}_2(t) = bx_1(t) + \overline{b} x_2(t)$$

$a$ and $b$ are complex numbers and $\overline{a}$ and $\overline{b}$ are their complex conjugates. These numbers can be written:

$$a = a_r + ja_i$$
$$\overline{a} = a_r - ja_i$$
$$b = b_r + jb_i$$
$$\overline{b} = b_r - jb_i$$

(a) First, assume that $\lambda = j$ in the equations for $x_1(t)$ and $x_2(t)$ above. Write out a system of differential equations using $\frac{d}{dt}\dot{x}_i(t)$ and $\dot{x}_i(t)$. (Just use coordinate changes this time. No need to work through direct substitutions unless you want to check your work that way.)

(b) How do the initial conditions for $x_i(t)$ translate into the initial conditions for $\dot{x}_i(t)$?
(c) Find the eigenvalues $\lambda_1$, $\lambda_2$ and associated eigenspaces for the differential equation matrix for $\ddot{x}_i(t)$ above.

(d) Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $z_{\lambda_1}(t)$, $z_{\lambda_2}(t)$. (These variables are in eigenbasis-aligned coordinates.)

(e) Solve the differential equation for $z_{\lambda_i}(t)$ in the eigenbasis.

(f) Convert your solution back into the $\ddot{x}_i(t)$ coordinates to find $\ddot{x}_i(t)$.

(g) Repeat the above for general complex $\lambda$.

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