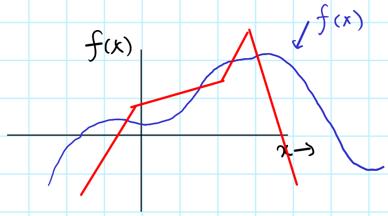
LEC 7A : LINEARIZATION

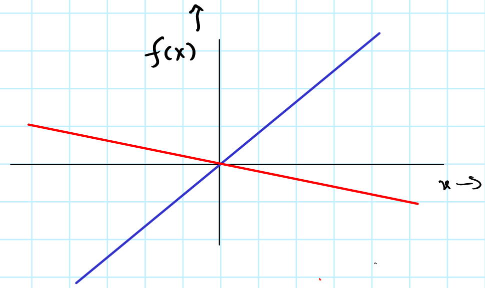
1) LINEARITY (FOR SCALAR FUNCTIONS)

→ GIVEN $y = f(x)$

→ PICTORIALLY :



NONLINEAR



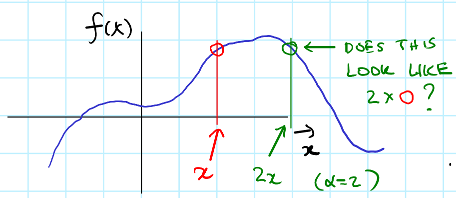
LINEAR

→ ALGEBRAICALLY : $f(x)$ is LINEAR if and only if (iff) it satisfies BOTH :

→ SCALING : GIVEN any α, and any x, $f(\alpha x) = \alpha f(x)$

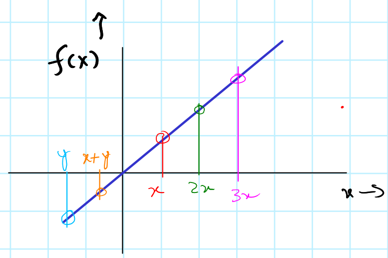
→ SUPERPOSITION : GIVEN any x and any y, $f(x+y) = f(x) + f(y)$

→ EXAMPLE : WHY IS



NOT LINEAR ?

→ EXAMPLE : WHY IS

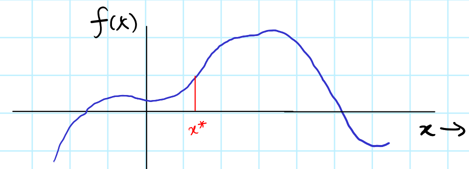


LINEAR ?

→ NOTE : TO SHOW THAT $f(x)$ IS NONLINEAR, YOU NEED FIND ONLY ONE α and ONE x/y WHERE SCALING OR SUPERPOSITION DOES NOT WORK. BUT TO SHOW THAT IT IS LINEAR, YOU HAVE TO PROVE IT FOR EVERY $\alpha/x/y$. THIS IS USUALLY ONLY POSSIBLE ALGEBRAICALLY.

2) LINEARIZATION OF A SCALAR FUNCTION

→ GIVEN (NONLINEAR) $f(x)$

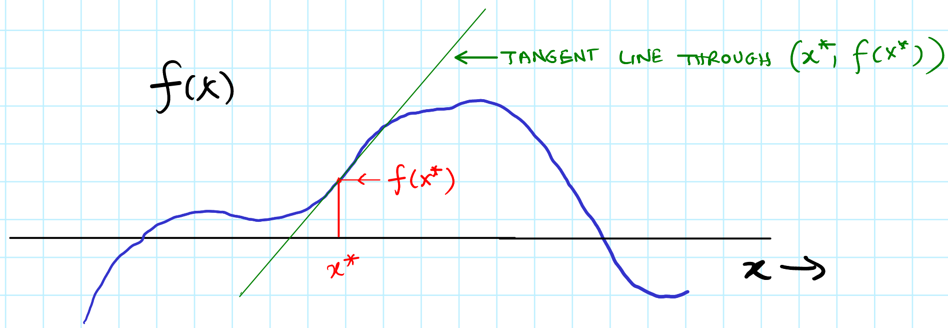


→ AND SOME SPECIFIC VALUE OF x, CALL IT x^* ("EXPANSION POINT")

→ THE "LINEARIZATION OF $f(x)$ ABOUT x^*" MEANS:

→ THE STRAIGHT LINE GOING THROUGH $(x^*, f(x^*))$, WITH SLOPE = $\left.\frac{df}{dx}\right|_{x^*}$

→ i.e., THE TANGENT LINE AT x^*

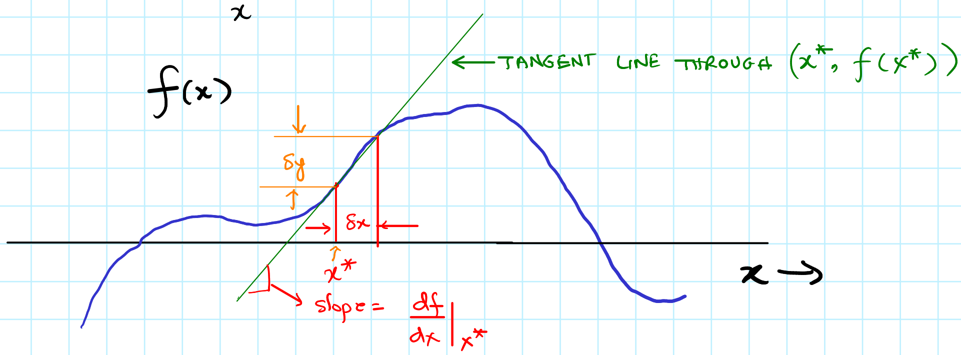


→ ALGEBRAICALLY:

$$f_L(\underbrace{x^* + \delta x}_{x = x^* + \delta x}) \triangleq f(x^*) + m\,\delta x$$

$m = \left.\frac{df}{dx}\right|_{x^*}$ = SLOPE of x^*

or $$\delta y \triangleq f_L(\underbrace{x^* + \delta x}_{x}) - f(x^*) = m\,\delta x$$



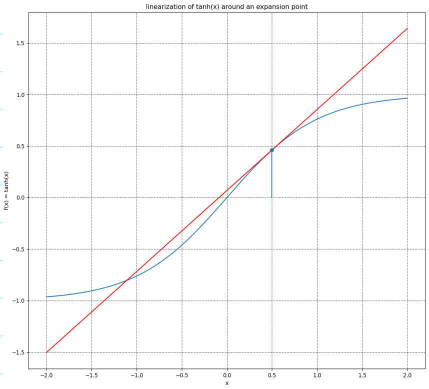
→ NOTE THAT δy IS A LINEAR FUNCTION OF δx (HENCE THE TERM LINEARIZATION)

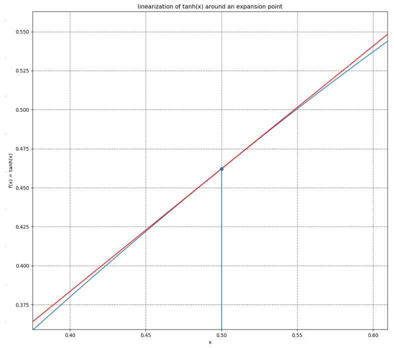
→ WHY IS LINEARIZATION RELEVANT?

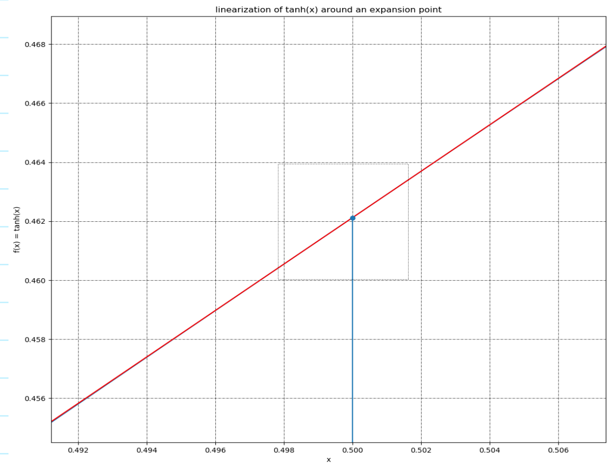
→ BECAUSE $f_L(x)$ APPROXIMATES $f(x)$, AND THE APPROXIMATION GETS BETTER AND BETTER AS x GETS CLOSER TO x^* (= as δx BECOMES SMALLER AND SMALLER)

→ MORE PRECISELY: THE RELATIVE ERROR $\boxed{\dfrac{f(x) - f_L(x)}{x - x^*}} \to 0$ as $x \to x^*$ ⟵ $\boxed{\text{DETAILS: DIS + HW}}$

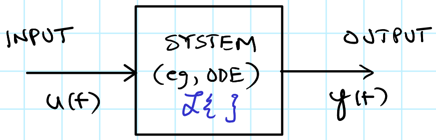
→ **INTUITIVELY:** AS YOU BLOW UP $f(x)$ around x^*, IT LOOKS MORE AND MORE LIKE ITS TANGENT LINE, WHICH IS EXACTLY WHAT $f_L(x)$ IS.







3) LINEARITY / NONLINEARITY FOR A SYSTEM (WITH I/O)



"FUNCTIONAL"

FUNCTION

→ DENOTE $y(t) = \mathcal{L}\{u(t)\}$ ← different from $y(t) = L(u(t))$

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→ FUNCTION: $y(t) = L(x(t))$. Given any <u>ONE value</u> of t, you can get $y(t)$ if you know $u(t)$

→ 1. Pick a value for t: eg, $t = 1.5$

→ 2. FIND $u(t)$: eg, $u(1.5) = 5$

→ 3. evaluate the function $L(\cdot)$ with this argument:

→ say $L(z) = z^2$, then $L(5) = 25$

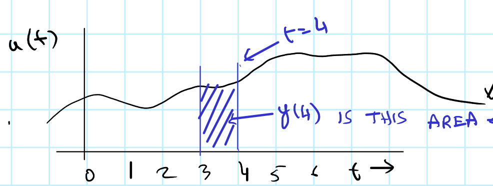
→ 4. So we have $y(t=1.5) = 25$.

→ <u>FUNCTIONAL</u>: $y(t) = \mathcal{L}\{u(t)\}$ ← VERY DIFFERENT FROM FUNCTION

→ YOU NEED THE <u>ENTIRE WAVEFORM</u> FOR $u(t)$, $\forall t$, BEFORE YOU CAN <u>FIND $y(t)$ FOR EVEN A SINGLE time point.</u>

→ JUST KNOWING $u(t)$ for one VALUE OF t IS <u>NOT ENOUGH</u> TO FIND $y(t)$, EVEN AT THE <u>SAME VALUE</u> of t. ← FUNCTIONALS CAN CAPTURE "MEMORY" IN A SYSTEM

→ <u>EXAMPLE</u> $y(t) = \mathcal{L}\{u(t)\} \triangleq \int_{t-1}^{t} u(\tau)\, d\tau$



$y(4)$ IS THIS AREA ← IT DEPENDS ON ALL THE VALUES OF $u(t)$ BETWEEN $t=3$ and $t=4$ → MEMORY

→ LINEARITY FOR SYSTEMS:

→ A SYSTEM IS LINEAR IFF IT SATISFIES BOTH SCALING AND SUPERPOSITION:

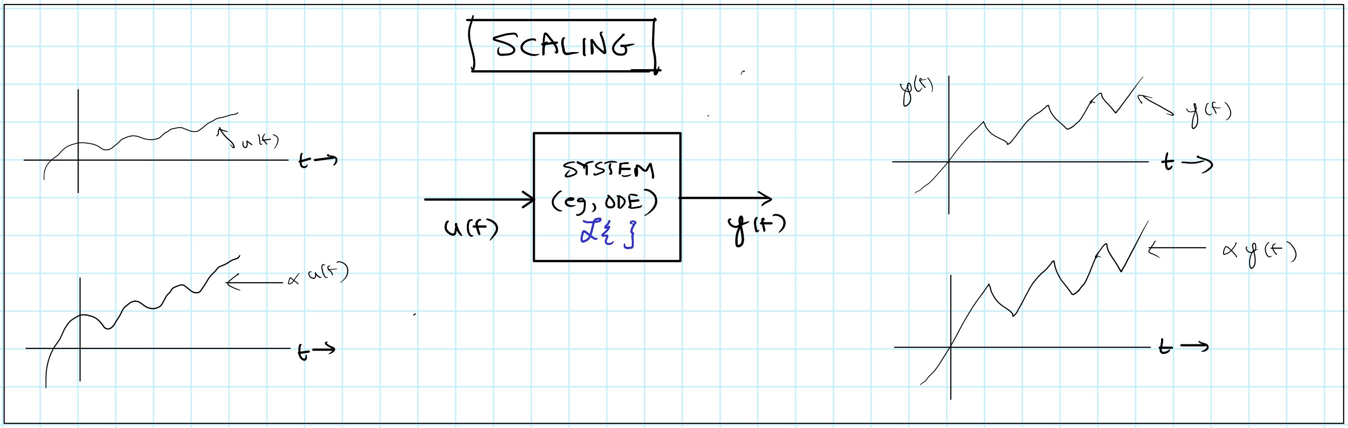
→ SCALING: $\mathcal{L}\{\alpha u(t)\} = \alpha \mathcal{L}\{u(t)\} \quad \forall \alpha, \forall u(t)$

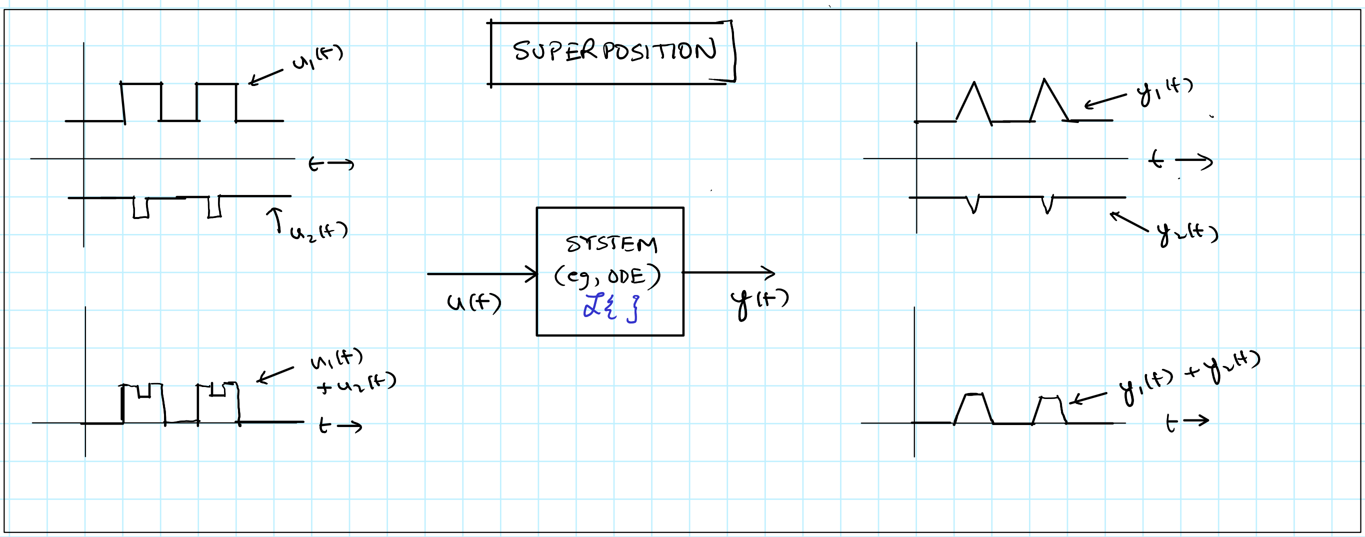
→ NOTE: α IS A CONSTANT (NOT A FUNCTION OF TIME)

→ SUPERPOSITION: $\mathcal{L}\{u_1(t) + u_2(t)\} = \mathcal{L}\{u_1(t)\} + \mathcal{L}\{u_2(t)\} \quad \forall u_1(t), u_2(t)$

→ NOW YOU SEE THAT LINEARITY for $f(x)$ WAS JUST A SIMPLE SPECIAL CASE OF THIS.

→ GRAPHICALLY (MUST HOLD FOR EVERY CHOICE OF $u(t), \alpha, u_1(t), u_2(t)$)





→ EXAMPLE: Given $\frac{dx}{dt} = f(x) + b\,u(t)$; all scalar. $u(t)$ = the input, $x(t)$ = the output.

→ Suppose $f(x)$ is linear. Is the ODE system linear?

→ TEST SCALING: IS IT TRUE THAT for any $u(t)$, any α, if you input $\alpha u(t)$, then you get $\alpha x(t)$ as output, where $x(t)$ is the output if you input $u(t)$?

→ I.e, does $(\alpha u(t), \alpha x(t))$ satisfy the ODE?

→ Suppose $y(t)$ is the output if you input $\alpha u(t)$: $\frac{d}{dt} y(t) = f(y(t)) + b\alpha u(t)$

→ Is $y(t) = \alpha x(t)$? Try it: $\frac{d}{dt}(\alpha x(t)) = \alpha \frac{d}{dt} x(t) = \alpha[f(x(t)) + b\,u(t)]$

→ Since $f(x)$ is linear, $\alpha f(x(t)) = f(\alpha x(t)) \Longrightarrow \frac{d}{dt}(\alpha x(t)) = f(\alpha x(t)) + b(\alpha u(t))$

↑ SO YES, ODE IS LINEAR

→ SUPERPOSITION ← DISCUSSION / HW

→ if $f(x)$ is NOT LINEAR, then $\dot{x} = f(x) + bu(t)$ is also not linear. ← DISCUSSION / HW.

4) LINEARIZATION OF AN ODE SYSTEM (SCALAR) ABOUT AN OPERATING PT
FIXED PT.
EQUILIBRIUM
DC OPERATING POINT

→ GIVEN $\frac{dx}{dt} = f(x) + bu(t)$, $f(x)$ is nonlinear. (4-1)

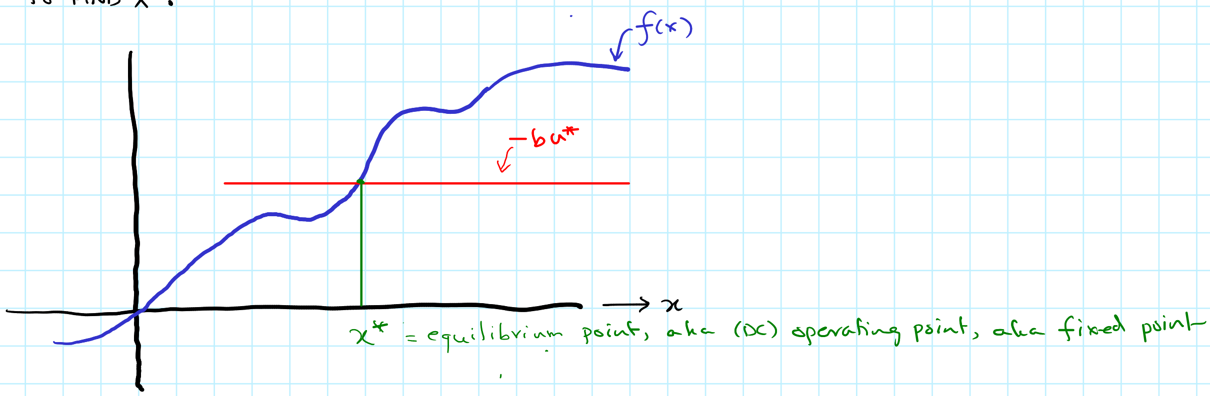
(OR, YOU ARE GIVEN)

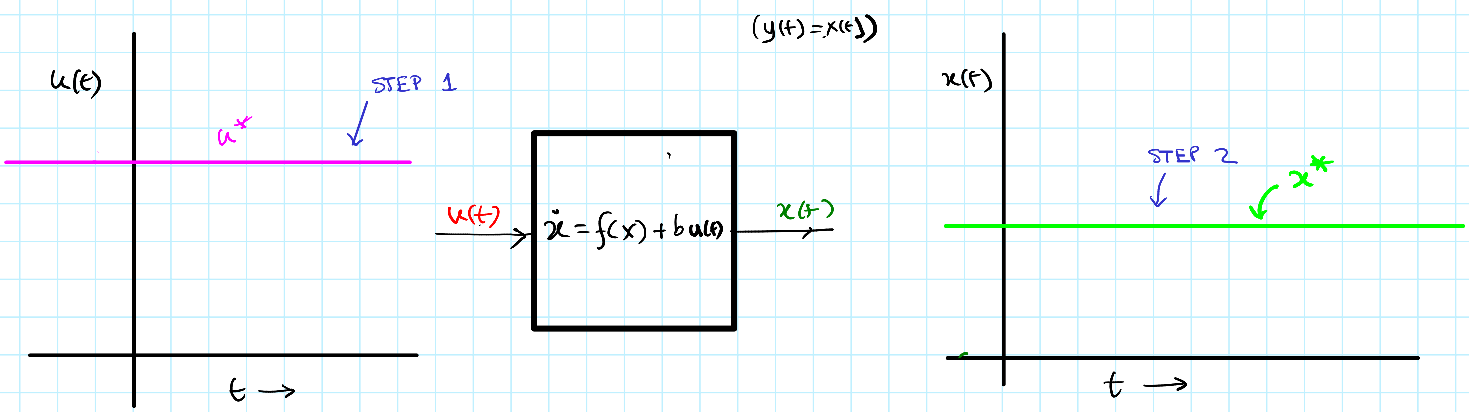
→ STEP 1 : CHOOSE A DC INPUT $u(t) \equiv u^*$, CONSTANT WITH TIME.

→ STEP 2 : SOLVE THE ODE for a DC SOLUTION $x(t) \equiv x^*$. I.E., SOLVE

$$f(x^*) = -bu^* \qquad (4-1)$$

TO FIND x^*.





→ STEP 3 : DEFINE $\delta x(t) \triangleq x(t) - x^*$, $\delta u(t) = u(t) - u^*$ and re-express the ODE (4-0) using $\delta x(t)$ and $\delta u(t)$.

$$\frac{d}{dt}\left[x^* + \delta x(t)\right] = f(x^* + \delta x(t)) + b(u^* + \delta u(t))$$

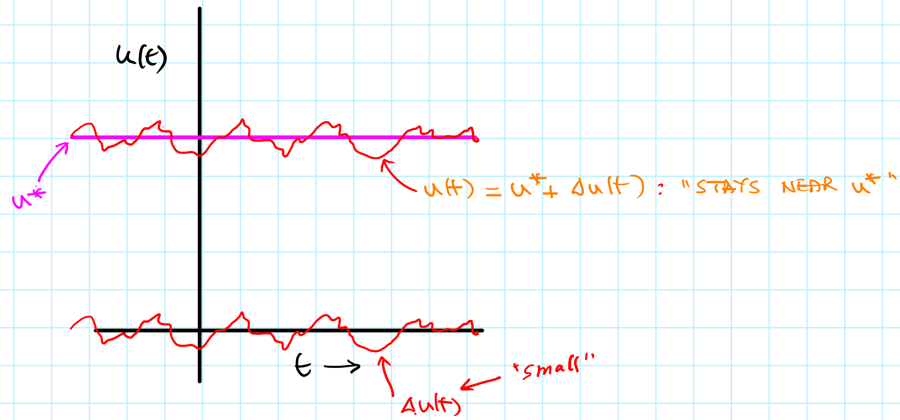
$$\Longrightarrow \frac{d}{dt}\delta x(t) = f(x^* + \delta x(t)) + bu^* + b\,\delta u(t) \qquad (4-2)$$

NOTE : $\delta u(t)$ IS OFTEN CALLED THE INPUT PERTURBATION (about u^*), and

$\delta y(t)$ " " " " OUTPUT DEVIATION (about x^*)

→ **STEP 4**: SUPPOSE YOU ARE <u>GIVEN</u> THAT $\delta u(t)$ is "SMALL".

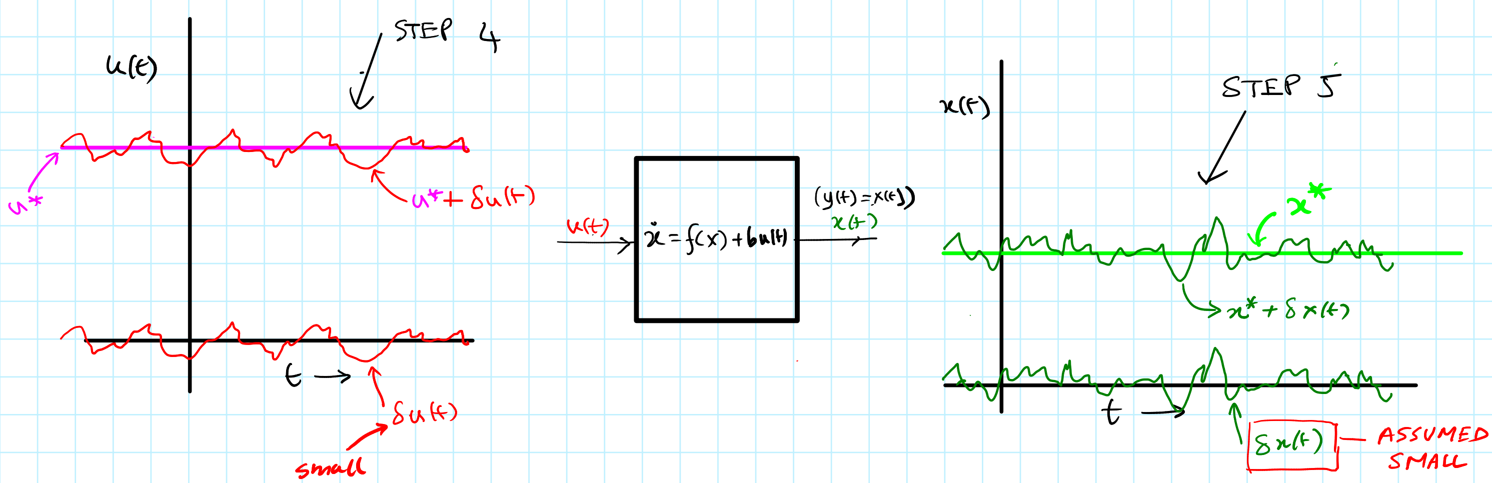
→ $u(t)$ is the input; you might be able to arrange for $u(t)$ to stay near u^*.



→ **STEP 5**: $\boxed{\text{ASSUME}}$ that $\delta x(t)$ (WHICH THE SYSTEM PRODUCES BY ITSELF, IN RESPONSE TO $\delta u(t)$) <u>ALSO IS SMALL</u>.

DISCUSSION.

→ THIS IS A BIG <u>ASSUMPTION</u>. IT IS VERY! POSSIBLE FOR IT NOT TO BE TRUE.



→ **STEP 6**: UNDER THIS ASSUMPTION (i.e., that $\delta x(t)$ is small (enough)), linearizing $f(\cdot)$ as

$$f(x^* + \delta x(t)) \simeq f(x^*) + \underbrace{m}_{\left.\frac{df}{dx}\right|_{x^*}} \delta x(t) \qquad (4\text{-}3)$$

is a reasonable approximation. Use it in (4-2).

<u>NOTE</u>: $\delta x(t)$ "small" <u>MEANS</u> small enough that (4-3) is a good approximation.

→ STEP 6 (CONTINUED): PUTTING (4-3) IN (4-2)

→ $$\frac{d}{dt}\delta x(t) = f(x^* + \delta x(t)) + bu^* + b\,\delta u(t)$$

$$\simeq f(x^*) + m\,\delta x(t) + bu^* + b\,\delta u(t)$$

→ SINCE x^* IS THE OPERATING POINT, DEFINED BY (4-1), we have: $f(x^*) + bu^* = 0$

→ ⟹ $$\boxed{\frac{d}{dt}\delta x(t) = m\,\delta x(t) + b\,\delta u(t)} \qquad (4\text{-}4).$$

→ THIS IS CALLED THE LINEARIZATION OF (4-0) ABOUT ITS DC OP. PT. x^* (and DC input u^*).

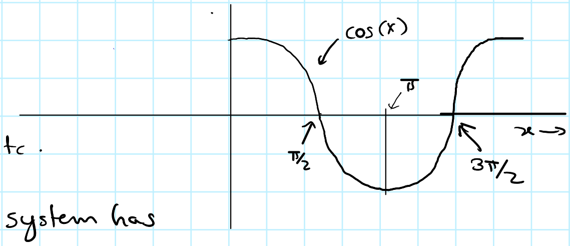
→ IT IS A LINEAR SYSTEM with input $\delta u(t)$ and output $\delta x(t)$.

→ IT IS TYPICALLY MUCH EASIER TO SOLVE THAN (4-0), WHICH IS NONLINEAR.

→ SOLVING IT CAN VALIDATE (OR INVALIDATE) THE ASSUMPTION IN STEP 5, THAT $\delta x(t)$ IS SMALL ← DISCUSSION

→ WHEN VALID, IT CAN PROVIDE TREMENDOUS INSIGHT INTO THE BEHAVIOUR OF THE ORIGINAL NONLINEAR ODE.

→ **EXAMPLE:** $\dfrac{dx}{dt} = f(x) + u(t)$, where $f(x) = \cos(x)$



→ choose $u^* = 0 \implies \cos(x^*) = 0 \implies x^* = \pi/2$
or $3\pi/2$, or $-\pi/2$, etc.

→ i.e., for the same DC input u^*, this system has multiple operating points!

DISCUSSION.

→ Suppose we take $x^* = \pi/2$, then $f(x^* + \delta x) \simeq -\delta x$, if δx is small

→ SYSTEM LINEARIZATION: $\boxed{\dfrac{d}{dt}\,\delta x(t) = -\delta x(t) + \delta u(t)}$

→ BY SOLVING THIS SYSTEM, IT CAN BE SHOWN THAT if $\delta u(t)$ is small, $\delta x(t)$ will also be correspondingly small.

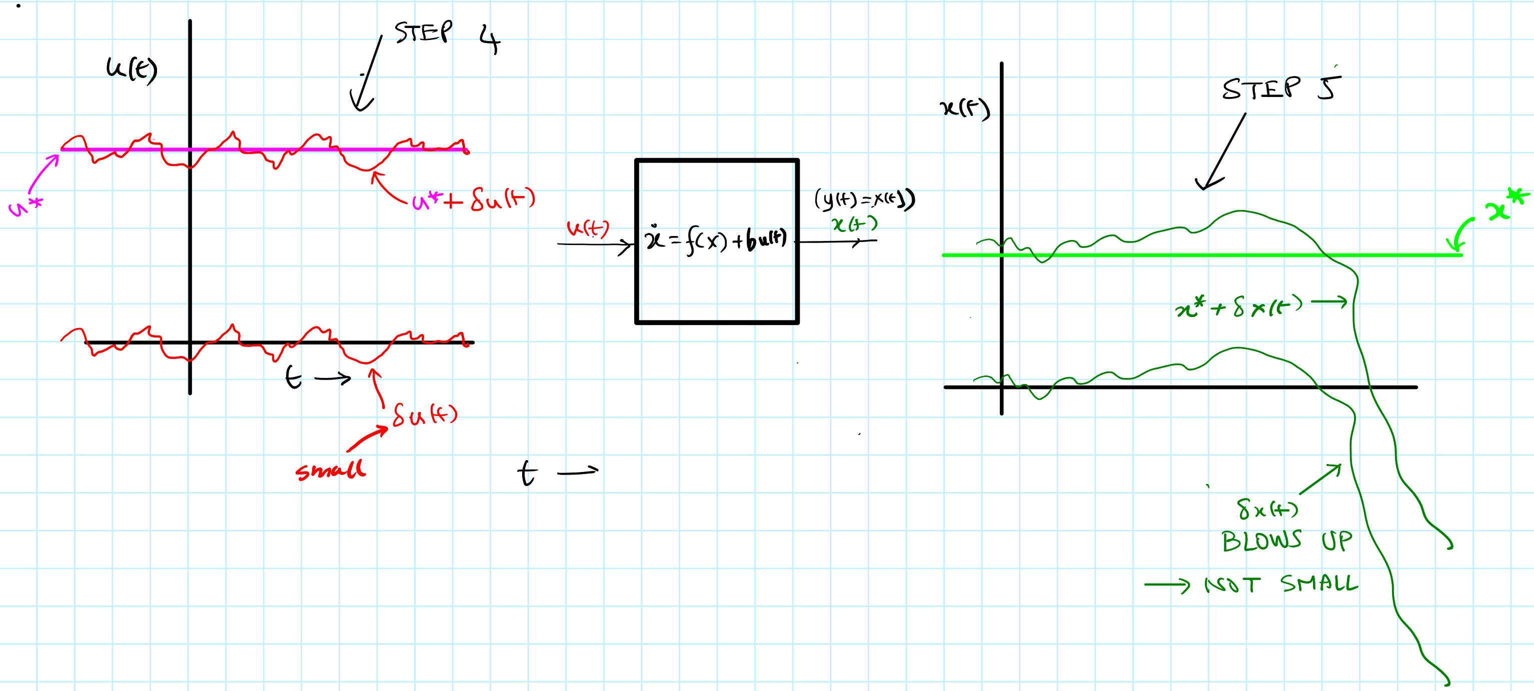
→ So STEP 5's ASSUMPTION ($\delta x(t)$ small) is indeed valid!

→ But if you take $x^* = 3\pi/2$, then the linearization becomes:

$\boxed{\dfrac{d}{dt}\,\delta x(t) = +\,\delta x(t) + \delta u(t)}$, for which the solution blows up up unboundedly, in general, even if $\delta u(t)$ is small.

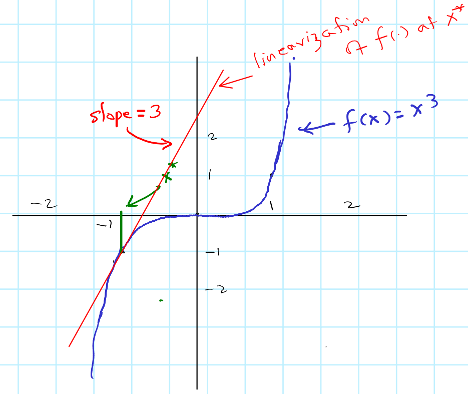
→ So STEP 5's ASSUMPTION ($\delta x(t)$ small) is INVALID, and subsequent steps are wrong! I.e., STEP 5's assumption led to a contradiction.

→ We should not use the linearization; it will not approximate the original nonlinear system well, even for small $\delta u(t)$.
— except during short periods of time when the assumption might hold!



→ EXAMPLE:

→ $\dfrac{dx}{dt} = f(x) + u(t)$, where $f(x) = x^3$



→ Choose $u^* = 1 \Rightarrow x^{*3} + 1 = 0 \Rightarrow x^* = -1$

→ $\left.\dfrac{df}{dx}\right|_{x^*} = 3x^{*2} = 3 = m$

$\Longrightarrow f(x^* + \delta x) \simeq \underset{f(x^*)}{\underset{\uparrow}{-1}} + 3\delta x$

→ SYSTEM LINEARIZATION: $\boxed{\dfrac{d}{dt}\delta x(t) = 3\,\delta x(t) + \delta u(t)}$

→ SOLUTION: even with no input perturbation ($\delta u(t) \equiv 0$), but with any non-zero IC $\delta x(0)$, the solution increases without bound!

→ $\delta x(t) = \delta x(0)\, e^{3t}$

$\Longrightarrow$ $\delta x(t)$ becomes arbitrarily large $\Longrightarrow$ STEP 5 ASSUMPTION INVALID!