→ WHAT WE DID LAST TIME

→ CIRCUIT ANALYSIS W SINUSOIDAL voltages/currents:

→ write sinusoids using complex number + conjugate

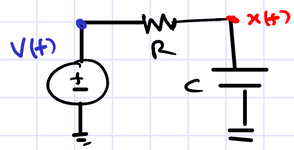
$\tilde{X} e^{j\omega t} + \bar{\tilde{X}} e^{-j\omega t}$

PHASOR

→ PHASORS make CAPS look like resistors $\tilde{I} = \dfrac{\tilde{V}}{Z_C} \rightarrow (1/j\omega C)$

→ PHASORS GREATLY SIMPLIFY CIRCUIT ANALYSIS (IF ALL ARE SINUSOIDAL)

→ STARTED LOOKING AT



→ TODAY:

→ CONTINUE WITH IN MORE DETAIL

→ TRANSFER FUNCTIONS : nice way of capturing how x(t) & V(t) are related.

→ PLOTTING PHASORS and TRANSFER FUNCTIONS

↳ LOG-LOG PLOTS look VERY NICE → WHY?

↳ BODE PLOTS

→ COMPLEX NUMBER ~~M~~-TUT CONTD.

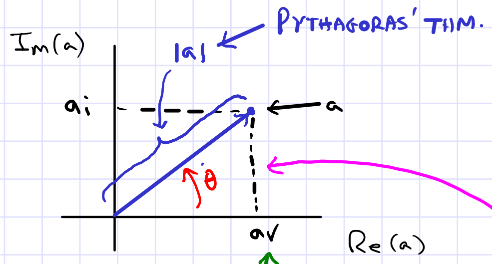
→ MAGNITUDE (OR ABSOLUTE VALUE) OF COMPLEX NUMBER:

→ $a = a_r + ja_i$ $|a| = \sqrt{a_r^2 + a_i^2}$

MAG. of a

→ NOTE: $|a|^2 = a\bar{a}$

→ GRAPHICAL DEPICTION OF COMPLEX NO:



→ θ = "angle of a" = $\angle a$

→ $\tan\theta = \dfrac{a_i}{a_r} \Rightarrow \boxed{\theta = \tan^{-1}\left(\dfrac{a_i}{a_r}\right)}$

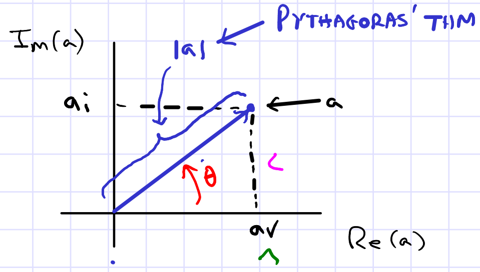
→ also: $\boxed{\sin(\theta) = \dfrac{a_i}{|a|}, \quad \cos(\theta) = \dfrac{a_r}{|a|}}$

→ NOTE: $\boxed{a = |a| e^{j\theta}}$ $= |a|\cos\theta + j|a|\sin(\theta)$

POLAR REPRESENTATION

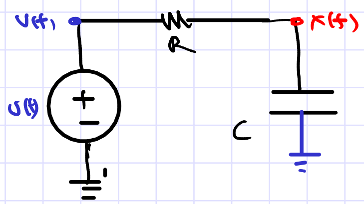
→ $\underbrace{a = a_r + j a_i}_{\text{CARTESIAN REPR.}} = \underbrace{M e^{j\theta}}_{\text{POLAR REPR.}}$ ← phase $= \angle a$

M → magnitude $= |a|$



→ NOTE : $\overline{(M e^{j\theta})} = M e^{-j\theta}$

→ SIMPLE RC CKT : PHASOR ANALYSIS

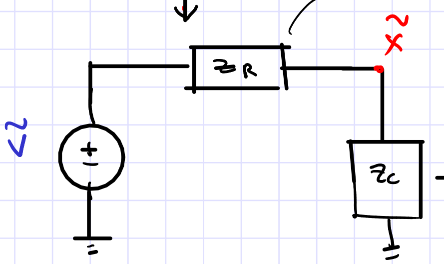


$v(t) = \tilde{V} e^{j\omega t}$ + c.c term → $\bar{\tilde{V}} e^{-j\omega t}$

known (eg: $\tilde{V} = \frac{1}{2j}$ if $v(t) = \sin(\omega t)$))

→ ASSUME $x(t)$ = sinusoid $= \tilde{X} e^{j\omega t}$ + c.c.t.

$\tilde{X}$: unknown: want to find.



→ $Z_R = R$

→ $Z_c = \frac{1}{j\omega c}$

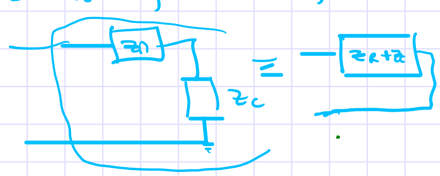
→ THIS IS A CKT DIAGRAM IN TERMS OF PHASORS ("PHASOR DOMAIN")
→ MAKES SENSE BECAUSE PHASORS OBEY KCL & KVL → DISCUSSION Problem 4

→ SOLVING THE CIRCUIT:

→ KCL @ x : $\frac{\tilde{X}}{Z_c} = \frac{\tilde{V} - \tilde{X}}{Z_R} \Rightarrow \tilde{X}(Z_c + Z_R) = \tilde{V} Z_c \Rightarrow \boxed{\tilde{X} = \tilde{V} \frac{Z_c}{Z_c + Z_R}}$

↑ IMPEDANCE DIVIDER FORMULA

→ Series impedance of



→ $\tilde{X} = \tilde{V} \frac{Z_c}{Z_c + Z_R} = \tilde{V} \frac{\frac{1}{j\omega c}}{\frac{1}{j\omega c} + R} = \frac{1}{1 + j\omega Rc}$

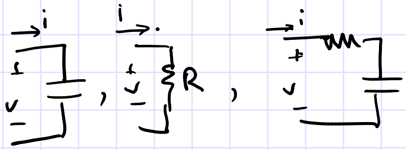
$$\boxed{\tilde{X} = \tilde{V} \frac{1}{1 + j\omega Rc}}$$

→ call it $H(\omega)$

FREQ. DOMAIN TRANSFER FUNCTION
(from $\tilde{V}$ to $\tilde{X}$)

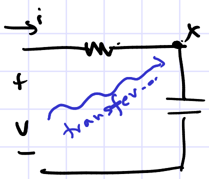
→ TRANSFER FUNCTION: GENERALIZATION OF IMPEDANCE/RESISTANCE:

→ $Z = \frac{\tilde{V}}{\tilde{I}}$



(PHASORS OF)

→

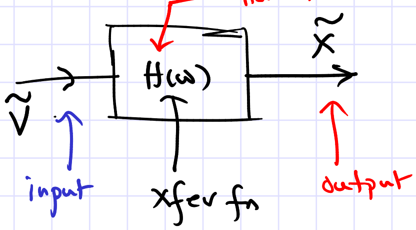


→ RATIO OF ANY TWO CKT QUANTITIES

→ voltages/currents (charge, flux, ...)

→ HELPS YOU LOOK AT A CIRCUIT AS A SYSTEM:

HOW YOU IMPLEMENT GETTING THIS IS A CIRCUIT PROBLEM

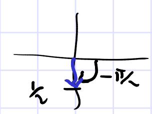


⟸ THIS MEANS $\tilde{X} = H(\omega)\cdot\tilde{V}$

→ NOW YOU SEE THAT: IMPEDANCES ARE ALSO A SPECIAL CASE OF TRANSFER FNS..

→ BACK TO: $\tilde{X} = \tilde{V}\,\frac{1}{1+j\omega RC}$ ← H(ω)

POLAR FORM →



→ Q: suppose $\tilde{V} = \frac{1}{2j} = \frac{-j}{2} = \frac{1}{2}e^{-j\pi/2}$ (i.e., $v(t) = \sin(\omega t)$), what is $x(t)$?)

→ $x(t) = \tilde{X}e^{j\omega t}$ + c.c. term, where $\tilde{X} = \frac{1}{2}e^{-j\pi/2} \times \frac{1}{1+j\omega RC}$ → H(ω)

Let's write H(ω) in POLAR FORM: $H(\omega) = M(\omega)\,e^{j\theta(\omega)}$

→ $M(\omega) = |H(\omega)|$, $\theta(\omega) = \angle H(\omega)$

⟹ $x(t) = \frac{1}{2}e^{-j\pi/2}\,M(\omega)\,e^{j(\omega t+\theta(\omega))}$ + c.c. term

$= \frac{1}{2}M(\omega)\,e^{j(\omega t+\theta(\omega)-\pi/2)}$ + c.c. term → $\frac{1}{2}M(\omega)\,e^{-j(\omega t+\theta(\omega)-\pi/2)}$

real and ≥0

$\cos(\phi-\pi/2) = \cos(\pi/2-\phi) = \sin(\phi)$

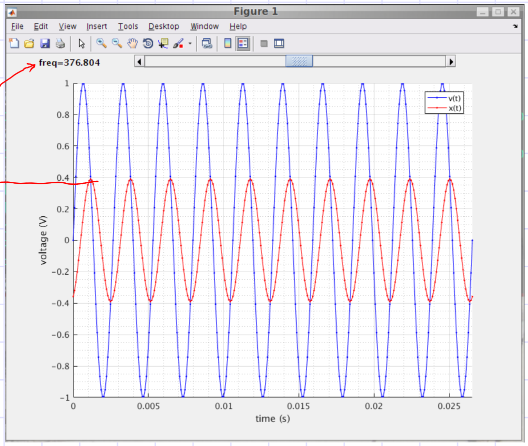
$x(t) = M(\omega)\cos(\omega t+\theta(\omega)-\pi/2)$

→ $\boxed{x(t) = \underbrace{M(\omega)}_{\text{amplitude}}\sin(\omega t + \underbrace{\theta(\omega)}_{\text{phase shift}})}$

$M(\omega) = \left|\frac{1}{1+j\omega RC}\right| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$

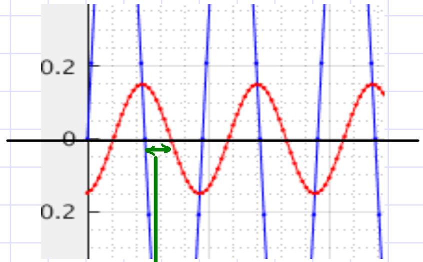
$\theta(\omega) = \angle\frac{1}{1+j\omega RC} = \angle\frac{1-j\omega RC}{1+\omega^2R^2C^2} = -\tan^{-1}(\omega RC)$

→ DEMO: v(t) vs. x(t) as f changes ($\omega = 2\pi f$)



$M(\omega = 2\pi \times$ ...

$$\frac{1}{\sqrt{1+\omega^2 R^2 C^2}}$$



$\theta(\omega) = -\tan^{-1}(\omega RC)$

→ 0 at $\omega = 0$

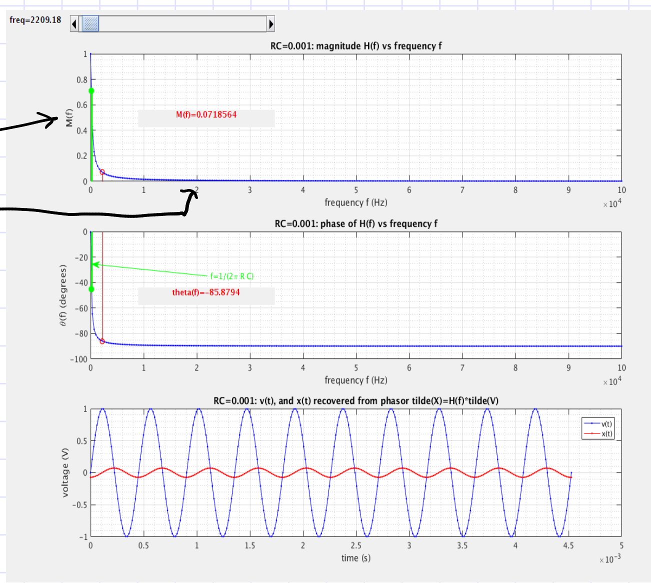
→ $-90° = -\pi/2$ radians at $\omega \to \infty$

→ VERY CLEAR THAT $M(\omega)$ & $\theta(\omega)$ TOTALLY DESCRIBE $x(t)$

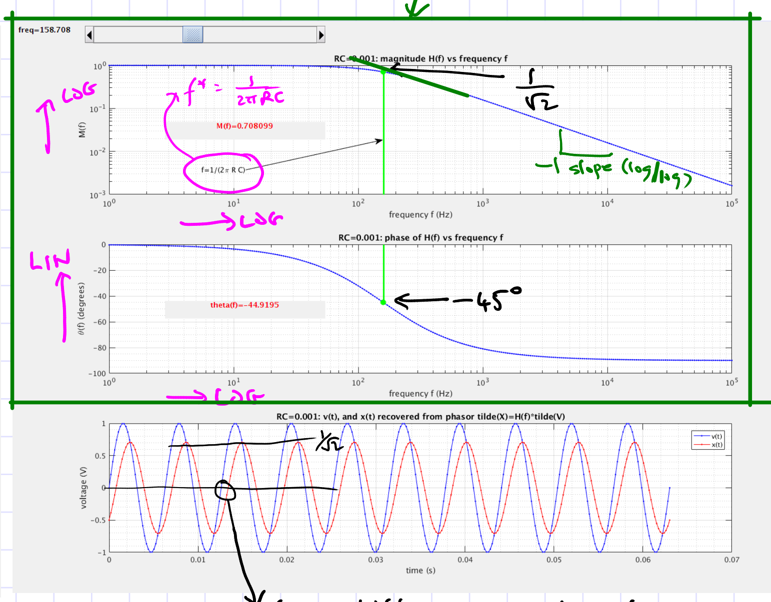
→ WHY NOT PLOT THEM vs f, TO VISUALLY appreciate how they change?

→ TRANSFER FN PLOT DEMO

LINEAR-LINEAR PLOT

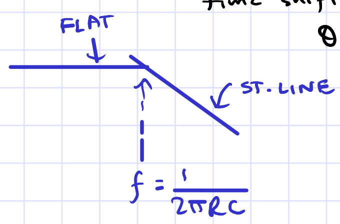


BODE PLOT of $H(2\pi f)$



time shift corresponding to $\theta = -45°$

→ WHY IS THE MAGNITUDE PLOT SO NICE?



→ CAN WE UNDERSTAND THIS?

→ Recall : $H(\omega) = \dfrac{1}{1+j\omega RC} = \dfrac{1}{1+j2\pi f RC}$

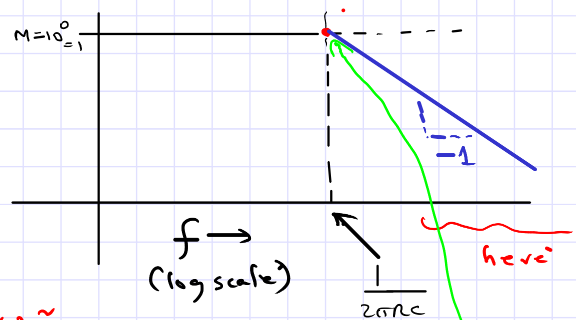
→ $M = |H(\omega)| = \dfrac{1}{\sqrt{1+4\pi^2 f^2 R^2 C^2}} = (1+4\pi^2 f^2 R^2 C^2)^{-1/2}$

→ what is $\log(M)$? ($\log_{10}$) $\quad \log(M) = -\frac{1}{2}\log(1+4\pi^2 f^2 R^2 C^2)$

→ consider values of f s.t. $1 \gg 4\pi^2 f^2 R^2 C^2 \Leftrightarrow \boxed{f \ll \frac{1}{2\pi RC}}$

→ then : $1+4\pi^2 f^2 R^2 C^2 \simeq 1$

→ $\log(M) \simeq -\frac{1}{2}\log(1) = 0$



→ now, try f s.t. $1 \ll 4\pi^2 f^2 R^2 C^2$

$\Rightarrow f \gg \frac{1}{2\pi RC}$

→ IN THIS REGIME : $1+4\pi^2 f^2 R^2 C^2 \simeq 4\pi^2 f^2 R^2 C^2$

$\Rightarrow \log(M) \simeq -\frac{1}{2}\log\left((2\pi f RC)^2\right)$

$= -1\times\log(2\pi f RC) = -\log(2\pi RC) - \log(f)$

$= +\log\left(\frac{1}{2\pi RC}\right) - \log(f)$

→ at $f = \frac{1}{2\pi RC}$, this is 0

→ thereafter, falls with a slope of -1

→ THE ABOVE ARE APPROXIMATIONS, VALID for $f \ll \frac{1}{2\pi RC}$ and $f \gg \frac{1}{2\pi RC}$

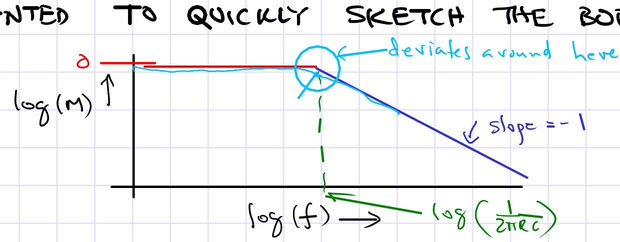
→ WHAT IS $H(2\pi f)$ EXACTLY at $f = \frac{1}{2\pi RC}$?

$$M = \frac{1}{\sqrt{1+(2\pi RC)^2 f^2}} = \frac{1}{\sqrt{1+(2\pi RC)^2 \times \frac{1}{(2\pi RC)^2}}} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} \;!$$

$\simeq 0.7071$

$$\theta = -\tan^{-1}(\omega RC) = -\tan^{-1}(2\pi RC f) = -\tan^{-1}(1) = -45°!$$

→ IF YOU WANTED TO QUICKLY SKETCH THE BODE PLOT OF $H(2\pi f) = \frac{1}{1+j\omega RC}$.



→ SUMMARY (PHASORS + TRANSFER FNS. + BODE PLOTS):

→ PHASORS: EASY CKT ANALYSIS FOR SINUSOIDAL SIGNALS!

→ RATIO OF ANY 2 CKT QUANTITIES: TRANSFER FUNCTION
→ USUALLY denoted as $H(\omega) = H(2\pi f)$

→ YOU CAN RECOVER TIME-DOMAIN WAVEFORMS EASILY FROM MAG/PHASE of $H(2\pi f)$

→ PLOT MAG/PHASE of $H(2\pi f)$ ON LOG-LOG AND LOG-LIN SCALE:
→ BODE PLOT.

→ MAG APPROXIMATED WELL WITH FLAT and -1 slope STRAIGHT LINE SEGMENTS.
$= -\frac{1}{\sqrt{2}}$ at $f = \frac{1}{2\pi RC}$

→ PHASE $= -45°$ at $f = \frac{1}{2\pi RC}$, 0 at $f=0$, → $-90°$ as $f \to \infty$

→ FROM PHASORS TO TIME-DOMAIN: GEOMETRICAL VIEW

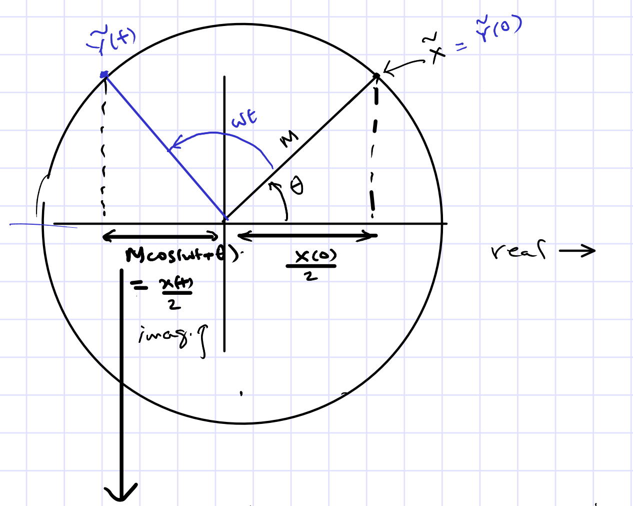
→ $x(t) = \tilde{X} e^{j\omega t}$ + c.c. term

$= M e^{j\theta} e^{j\omega t}$ + c.c. term

$= \underbrace{M e^{j(\omega t + \theta)}}_{\tilde{Y}(t)}$ + c.c. term

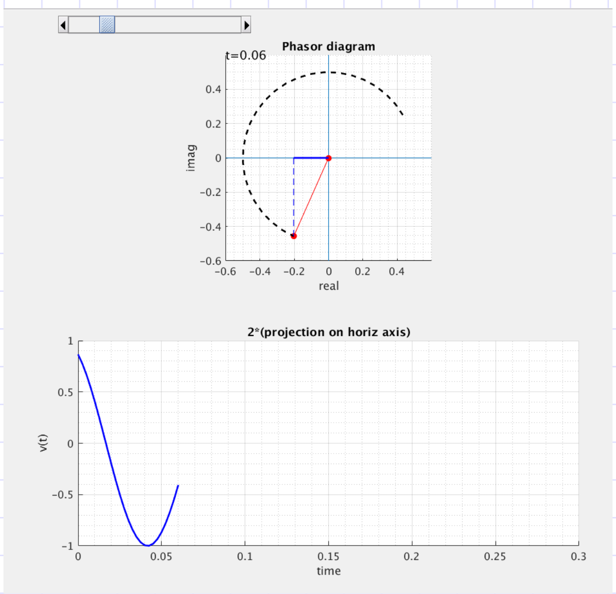
→ $x(t) = M e^{j(\omega t+\theta)} + M e^{-j(\omega t+\theta)}$

$= \underline{2M\cos(\omega t + \theta)}$

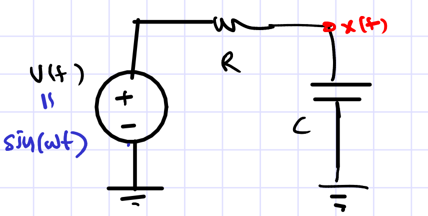


→ PROJECT $\tilde{Y}(t)$ onto horizontal axis
→ MULTIPLY BY 2
→ you've got $x(t)$!

→ SLIDER DEMO:

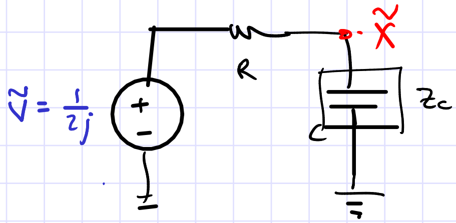


→ PHASOR SOLUTIONS vs DIFFERENTIAL EQN SOLUTIONS:



ODE: $\dfrac{dx}{dt} = \dfrac{v(t) - x(t)}{RC}$

→ can specify IC: $x(0) = x_0$

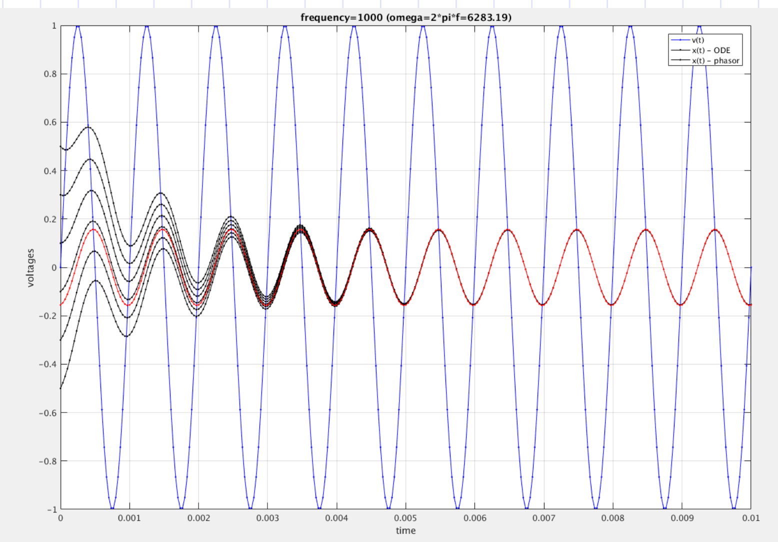


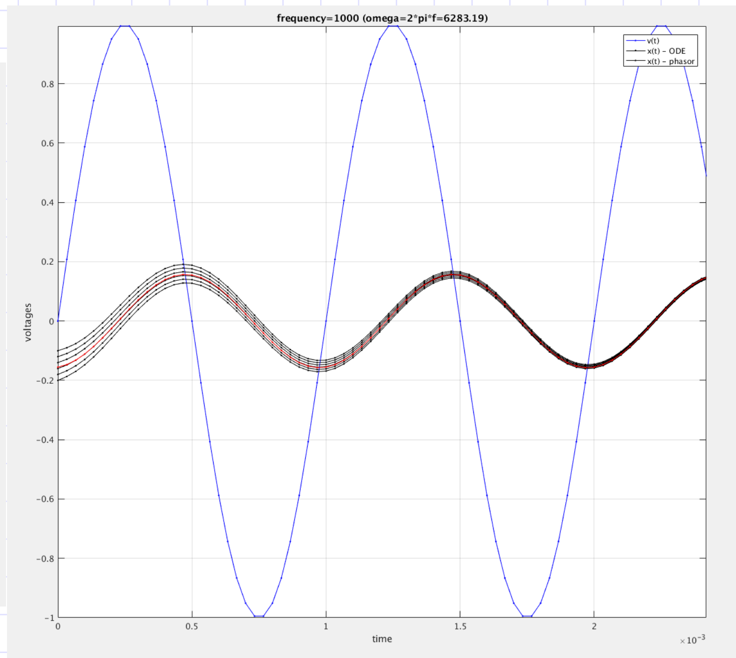
$$\tilde{X} = \tilde{V}\,\frac{1}{1 + j2\pi f RC} \Longrightarrow x(t) = M(f)\sin(2\pi f t + \theta(f))$$

→ THE QUESTION IS: IS x(t) from solving the ODE the same as x(t) ??

→ DEMO: ODE NUMERICAL SOLN vs PHASOR-derived solution.

↳ with various ICs.





→ PHASOR ANALYSIS AUTOMATICALLY FINDS THE IC THAT MAKES THE ODE SOLUTION A PERFECT SINE WAVE → NO "STARTUP TRANSIENTS"

STEINMETZ

BODE