→ WHAT WE DID LAST TIME

→ CIRCUIT ANALYSIS W/ SINEUSOIDAL VOLTAGES/CURRENTS:
   → write sinusoids using complex number + conjugate
     \[ x(t) + \overline{x}(t) e^{-j\omega t} \]

→ PHASORS make CAPS look like resistors
   \[ i = \frac{v}{R} \rightarrow (\frac{1}{j\omega C}) \]

→ PHASORS GREATLY SIMPLIFY CIRCUIT ANALYSIS (IF ALL ARE SINEUSOIDAL)

→ STARTED LOOKING AT

→ TODAY:

→ CONTINUE WITH IN MORE DETAIL

→ TRANSFER FUNCTIONS: nice way of capturing how INPs/LUTs are related

→ PLOTTING PHASORS AND TRANSFER FUNCTIONS
   → Log-log plots look very nice → why?
   → Bode plots

→ COMPLEX NUMBER MATCONT:

→ MAGNITUDE (OR ABSOLUTE VALUE) OF COMPLEX NUMBER:
   \[ |a| = \sqrt{a_x^2 + a_y^2} \]

→ NOTE: \((a)^* = a\overline{a}\)

→ GRAPHICAL DEPICTION OF COMPLEX NO:

→ \(\theta = \text{angle of } a = \angle a\)

→ \(\tan \theta = \frac{a_y}{a_x} \quad \Rightarrow \quad \theta = \tan^{-1} \left( \frac{a_y}{a_x} \right) \)

→ also:
   \[ \sin(\theta) = \frac{a_y}{|a|}, \quad \cos(\theta) = \frac{a_x}{|a|} \]

→ NOTE:
   \[ a = |a| e^{j\theta} \]

→ POLAR REPRESENTATION
\[ a = a_0 + j a_1 \]
\[ \text{mag} = |a| \]
\[ \text{phase} = \phi \]

**Cartesian Representation**

**Polar Representation**

\[ (Me^{j\theta}) = M e^{-j\theta} \]

**Simple RC Circuit: Phasor Analysis**

\[ V(t) = V_0 e^{j\omega t} + c.c. \text{ term} \]

**Assume**

\[ x(t) = \sin(\omega t) = X e^{j\omega t} + c.c.t. \]

**Unknown: Want to find**

\[ Z_L = R \]

\[ Z_C = \frac{1}{j\omega C} \]

**This is a circuit diagram in terms of phasors ("phasor domain")**

**Makes sense because phasors obey KCL & KVL**

**Discussion Problem 4**

**Solving the Circuit**

\[ KCL @ x: \]

\[ \frac{V}{Z_L} = \frac{V}{Z_C} \]

\[ X(\frac{Z_C + Z_R}{Z_R}) = \sqrt{V} Z_C \]

\[ X = \sqrt{\frac{V}{Z_C + Z_R}} \]

**Series Impedance Formula**

**Impedance Divider Formula**

\[ X = \frac{1}{\frac{j\omega C}{Z_C + Z_R}} \]

**Freq. Domain Transfer Function**

\[ X = \frac{1}{1 + j\omega RC} \]

\[ \text{Call it } H(j\omega) \]

\[ \text{Freq. Domain Transfer Function} \]
**TRANSFER FUNCTION**: GENERALIZATION OF IMPEDANCE/RESISTANCE:

\[ Z = \frac{j \omega}{s} \]

**Phasors of**

\[ \frac{-j \omega}{s} \]

**Ratio of any two C.E.T quantities**

\[ \frac{V_2}{V_1} \]

**Helps you look at a circuit as a system**:

\[ V_2 \rightarrow H(s) \rightarrow X \]

\[ x \rightarrow H(s) \rightarrow V \]

\[ \text{THIS MEANS } \quad x = H(s) \cdot v \]

**Now you see that: Impedances are also a special case of transfer Fns.**

\[ x = H(s) \cdot x \]

**Back to**:

\[ X = \frac{1}{1 + j \omega RC} \]

Polar form

\[ X = |X| \cdot \angle \theta \]

**Q: Suppose**

\[ \tilde{V} = \frac{1}{2j} \cdot \frac{1}{2} = \frac{1}{2} e^{-j \frac{\pi}{6}} \]

(i.e. \( v(t) = \sin(\omega t) \)), what is \( x(t) \) ?

\[ x(t) = X e^{j \omega t} \]

\[ x = \frac{1}{2} e^{-j \frac{\pi}{6}} \]

\[ H(s) = \frac{1}{1 + j \omega RC} \]

**Let's write** \( H(s) \) **in Polar Form**: \( H(s) = M(s) e^{j \theta(s)} \)

\[ M(s) = |H(s)|, \quad \theta(s) = \angle H(s) \]

\[ X(s) = M(s) \cdot e^{j \theta(s)} \]

\[ \text{Real and } \geq 0 \]

\[ x(t) = M(s) \cos(\omega t + \theta(s)) \]

\[ \theta(s) = \angle \frac{1}{1 + j \omega RC} = \angle \frac{(1 - j \omega RC)}{1 + j \omega RC} = -\tan^{-1}(\omega RC) \]

\[ M(s) = \frac{1}{|1 + j \omega RC|} = \frac{1}{\sqrt{1 + \omega^2 RC^2}} \]
DEMO: $v(t)$ vs. $x(t)$ as $f$ changes ($\omega = 2\pi f$)

$\theta(t) = -\tan^{-1}(\omega C)$

VERY CLEAR THAT $M(\omega)$ & $\theta(\omega)$ TOTALLY DESCRIBE $x(t)$

WHY NOT PLOT THEM VS $f$, TO VISUALLY APPRECIATE HOW THEY CHANGE?

TRANSFER FN PLOT DEMO

VERY CLEAR THAT $M(\omega)$ & $\theta(\omega)$ TOTALLY DESCRIBE $x(t)$

WHY NOT PLOT THEM VS $f$, TO VISUALLY APPRECIATE HOW THEY CHANGE?

Why is the magnitude plot so nice?

Can we understand this?
Recall: \( H(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j2\pi f RC} \)

\[ M = \int |H(\omega)| = \frac{1}{\sqrt{1 + 4\pi^2 f^2 RC^2}} = (1 + 4\pi^2 f^2 RC^2)^{-\frac{1}{2}} \]

\[ \log M \]

What is \( \log(M) \)?

\[ \log(M) = -\frac{1}{2} \log\left(1 + 4\pi^2 f^2 RC^2\right) \]

Consider values \( f \), s.t. \( 1 \gg 4\pi^2 f^2 RC^2 \)

Then:

\[ 1 + 4\pi^2 f^2 RC^2 \approx 1 \]

\[ \log(M) \approx -\frac{1}{2} \log(1) = 0 \]

Now, try \( f \) s.t. \( 1 \ll 4\pi^2 f^2 RC^2 \)

\[ f \gg \frac{1}{2\pi RC} \]

In this regime:

\[ 1 + 4\pi^2 f^2 RC^2 \approx 4\pi^2 f^2 RC^2 \]

\[ \log(M) \approx -\frac{1}{2} \log\left((2\pi f RC)^2\right) = -1 \times \log(2\pi f RC) = -\log(2\pi f RC) - \log(f) = + \log\left(\frac{1}{2\pi RC}\right) + \log(f) \]

At \( f = \frac{1}{2\pi RC} \), this is 0.

Therefore, falls with a slope of -1.

THE ABOVE ARE APPROXIMATIONS, VALID FOR \( f \ll \frac{1}{2\pi RC} \) AND \( f \gg \frac{1}{2\pi RC} \).

What is \( f(2\pi f) \) EXACTLY at \( f = \frac{1}{2\pi RC} \)?

\[ M = \frac{1}{\sqrt{1 + (2\pi RC)^2 f^2}} = \frac{1}{\sqrt{1 + 4\pi^2 RC^2 \frac{1}{(2\pi RC)^2}}} = \frac{1}{\sqrt{1 + 1}} = \frac{1}{\sqrt{2}} \approx 0.7071 \]

\[ \Theta = -\tan^{-1}(\omega RC) = -\tan^{-1}(2\pi RC f) = -\tan^{-1}(1) = -45^\circ \]

IF YOU WANTED TO QUICKLY SKETCH THE BODE PLOT OF \( H(\omega f) = \frac{1}{1 + j\omega RC} \).
**SUMMARY (PHASORS + TRANSFER FUNCS. + BODE PLOTS):**

**PHASORS: EASY CFT ANALYSIS FOR SINEWODAL SIGNALS!**

**RATIO OF ANY 2 CFT QUANTITIES: TRANSFER FUNCTION**

→ Usually denoted as \( H(f) = H(2\pi f) \)

**YOU CAN RECOVER TIME-DOMAIN WAVEFORMS EASILY FROM MAG/PHASE \( \frac{N_f}{H(2\pi f)} \)**

**PLOT MAG/PHASE \( \frac{N_f}{H(2\pi f)} \) ON LOG-LOG AND LOG-LIN SCALE:**

→ **BODE PLOT.**

→ **MAG APPROXIMATED WELL WITH FLAT AND \(-1\) SLOPE STRAIGHT LINE SEGMENTS.**

\[ \frac{N_f}{H(2\pi f)} = \frac{1}{\sqrt{2}} \quad \alpha = \frac{1}{2\pi f} \]

→ **PHASE**

\[ \begin{align*}
\phi &= -45^\circ \quad \alpha = \frac{1}{2\pi f} \\
\alpha &\to 0^\circ \quad \phi &\to \infty \\
\end{align*} \]

→ **FROM PHASORS TO TIME-DOMAIN: GEOMETRICAL VIEW**

\[ x(t) = \sum x(t) e^{j\omega t} + \text{c.c. term} \]

\[ = M e^{j(\omega t + \theta)} + \text{c.c. term} \]

** slider demo:**

\[ X(f) = M e^{j(\omega t + \theta)} + M e^{-j(\omega t + \theta)} \]

→ **MULTIPLY BY 2**

→ **YOU’VE GOT \( x(t) \)!**
→ PHASOR SOLUTIONS vs DIFFERENTIAL EQN SOLUTIONS:

\[ \frac{d}{dt} x(t) = \frac{v(t) - x(t)}{RC} \]

\[ X = \frac{V}{1 + j2\pi fRC} \rightarrow x(t) = M(f) \sin(2\pi ft + \phi(f)) \]

→ can specify IC: \( x(0) = x_0 \)

→ THE QUESTION IS: IS \( x(t) \) from solving the ODE the same as ??

→ DEMO: ODE NUMERICAL SOLN vs PHASOR-derived solution

→ with various ICs.

→ PHASOR ANALYSIS AUTOMATICALLY FINDS THE IC THAT

MAKES THE ODE SOLUTION A PERFECT SINE WAVE → NO "STARTUP TRANSIENTS"

STEINMETZ  BODE