1 Stability

Discrete time systems

A discrete time system is of the form:

$$\bar{x}[t + 1] = A\bar{x}[t] + B\bar{u}[t]$$

This system is stable if $|\lambda_i| < 1$ for all $\lambda_i$, where $\lambda_i$'s are the eigenvalues of $A$. If we plot all $\lambda_i$ for $A$ on the complex plane, if all $\lambda_i$ lie within (not on) the unit circle, then the system is stable.

Continuous time systems

A continuous time system is of the form:

$$\frac{d\bar{x}}{dt}(t) = A\bar{x}(t) + B\bar{u}(t)$$

This system is stable if $\text{Re}\{\lambda_i\} < 0$ for all $\lambda_i$, where $\lambda_i$'s are the eigenvalues of $A$. If we plot all $\lambda_i$ for $A$ on the complex plane, if all $\lambda_i$ lie to the left of $\text{Re}\{\lambda_i\} = 0$, then the system is stable.
2 Continuous time system responses

We have a system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ with eigenvalues $\lambda_i$. For each set of $\lambda_i$ values plotted on the complex plane, sketch $\mathbf{x}(t)$ with an initial condition of $x(0) = 1$. Determine if each system is stable.
Answer

Stable  Stable  Unstable

Unstable  Unstable
3  Discrete time system responses

We have a system $x[k + 1] = \lambda x[k]$. For each $\lambda$ value plotted on the real-imaginary axis, sketch $x[k]$ with an initial condition of $x[0] = 1$. Determine if each system is stable.

![Diagram of the complex plane with points A, B, C, D, E, F labeled along the real axis, and the imaginary axis labeled Im{\lambda} and Re{\lambda}.]

**Answer**

- **A** - Unstable
- **B** - Unstable
- **C** - Stable
4 Stability in continuous time system

Remember the spring-mass system introduced in Discussion 8A:

We assumed that each spring is linear with spring constant $k$ and resting length $X_0$. The differential equation modeling this system was $\frac{d^2 y}{dx^2} = -\frac{2k}{m} (y - X_0 \frac{y}{\sqrt{y^2 + a^2}})$. We built a state space model that describes how the displacement $y$ of the mass from the spring base evolves. The state variables were $x_1 = y$ and $x_2 = \dot{y}$. Then we linearized the model around the equilibrium point
\((x_1, x_2) = (0, 0)\), assuming \(X_0 < a\). The linearized model is presented below.

\[
\dot{x} = \begin{bmatrix}
0 & 1 \\
\frac{-2k}{m} \left(1 - \frac{X_0}{a}\right) & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}.
\]

Compute the eigenvalues of your linearized model. Is this equilibrium stable?

**Answer**

To compute the eigenvalues, we solve

\[
0 = \det(A - \lambda I) = \det \left( \begin{bmatrix}
-\lambda & 1 \\
\frac{-2k}{m} \left(1 - \frac{X_0}{a}\right) & -\lambda \\
\end{bmatrix} \right) = \lambda^2 + \frac{2k}{m} \left(1 - \frac{X_0}{a}\right).
\]

Since \(X_0 < a\), this means that \(1 - \frac{X_0}{a} > 0\). So we have a pair of imaginary eigenvalues

\[
\lambda = \pm \sqrt{\frac{2k}{m} \left(1 - \frac{X_0}{a}\right)}.
\]

Since the linearized system has purely imaginary eigenvalues that are not repeated, their real parts are zero. Therefore the equilibrium is unstable.

### 5 Stability in discrete time system

Determine which values of \(\alpha\) and \(\beta\) will make the following discrete-time state space models stable. Assume, \(b \neq 0\).

a) 

\[x[t + 1] = \alpha x[t] + bu(t)\]

**Answer**

\(|\alpha| < 1\)

b) 

\[
\bar{x}[t + 1] = \begin{bmatrix}
\alpha & -\beta \\
\beta & \alpha \\
\end{bmatrix} \bar{x}[t] + b\bar{u}(t)
\]
Answer
The eigenvalues of this system are:
\[ \lambda = a \pm j\beta \]
\[ |\lambda| = \sqrt{a^2 + \beta^2} \]
For this system to be stable, \(|\lambda| < 1\), so
\[ a^2 + \beta^2 < 1 \]

\[ \mathbf{x}[t + 1] = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mathbf{x}[t] + b\mathbf{i}(t) \]

Answer
The eigenvalues of this system are
\[ \lambda = 1, 1 \]
This means that regardless of \( a \), this system is always unstable.